

Set coverings and invertibility of Functional Galois Connections

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ABSTRACT. We consider equations of the form $Bf = g$, where B is a Galois connection between lattices of functions. This includes the case where B is the Legendre-Fenchel transform, or more generally a Moreau conjugacy. We characterise the existence and uniqueness of a solution f in terms of generalised subdifferentials. This extends a theorem of Vorobyev and Zimmermann, relating solutions of max-plus linear equations and set coverings. We give various illustrations.

1. Introduction

We call (dual) *functional Galois connection* a (dual) Galois connection between a sublattice \mathcal{F} of $\overline{\mathbb{R}}^Y$ and a sublattice \mathcal{G} of $\overline{\mathbb{R}}^X$, where X, Y are two sets and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, see Section 2 for definitions. An important example of functional Galois connection is the Legendre-Fenchel transform, and more generally, the Moreau conjugacy [Mor70] associated to a kernel $b : X \times Y \rightarrow \overline{\mathbb{R}}$,

$$(1) \quad B : \overline{\mathbb{R}}^Y \rightarrow \overline{\mathbb{R}}^X, \quad Bf(x) = \sup\{b(x, y) - f(y) \mid y \in Y\},$$

where $b(x, y) - f(y)$ is an abbreviation of $b(x, y) + (-f(y))$, with the convention that $-\infty$ is absorbing for addition, i.e., $-\infty + \lambda = \lambda + (-\infty) = -\infty$, for all $\lambda \in \overline{\mathbb{R}}$. Moreau conjugacies are instrumental in nonconvex duality, see [RW98, Chapter 11, Section E],[Sin97]. Max-plus linear operators with kernel, which are of the form $f \mapsto B(-f)$, arise in deterministic optimal control and asymptotics, and have been widely studied, see in particular [CG79, MS92, BCOQ92, KM97, Aki99, GM01]. Other examples of functional Galois connections include dualities for quasi-convex functions (see for instance [Sin97, Vol98, Sin02]).

We consider here general (dual) Galois connections between the set \mathcal{F} of lower semicontinuous functions from a Hausdorff topological space Y to $\overline{\mathbb{R}}$ and $\mathcal{G} = \overline{\mathbb{R}}^X$. Any functional Galois connection $B : \mathcal{F} \rightarrow \mathcal{G}$ has the form:

$$Bf(x) = \sup\{b(x, y, f(y)) \mid y \in Y\},$$

2000 *Mathematics Subject Classification.* Primary 06A15; Secondary 52A01, 44A05.

Key words and phrases. Galois Correspondence, Galois Connection, Moreau Conjugacy, Legendre-Fenchel Transform, Max-Plus Algebra, Abstract Duality, Abstract Convex Analysis, Subdifferential, Essentially Smooth Convex Function.

Date: July 26, 2003. *ESI-preprint:* January 28, 2004. *Revised:* March 25, 2004.

This work has been partially supported by the Erwin Schrödinger International Institute of Mathematical Physics (ESI).

where $b : X \times Y \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ is such that $b(x, \cdot, \alpha) \in \mathcal{F}$, for all $x \in X$ and $\alpha \in \overline{\mathbb{R}}$, and $b(x, y, \cdot)$ is nonincreasing, right continuous and sends $+\infty$ to $-\infty$, for all $x \in X$ and $y \in Y$, see Theorem 2.1 and Proposition 2.3 below. Representation theorems of this type have been obtained previously by Maslov and Kolokoltsov [KM87, Kol90, Kol92] and Martínez-Legaz and Singer [MLS90, Sin97], see Section 2.

Given a map $g \in \mathcal{G}$ and a functional Galois connection $B : \mathcal{F} \rightarrow \mathcal{G}$, we consider the problem:

$$(\mathcal{P}) : \quad \text{Find } f \in \mathcal{F} \text{ such that } Bf = g.$$

In particular, we look for effective conditions on g for the solution f to exist and be unique. When X, Y are finite sets, $\mathcal{F} = \mathbb{R}^Y$, $\mathcal{G} = \mathbb{R}^X$, and B is as in (1) with a real valued kernel b , the solutions of (\mathcal{P}) were characterised by Vorobyev [Vor67, Theorem 2.6] in terms of “minimal resolvent coverings” of X (in fact, Vorobyev considered equations of the type $\min_{y \in Y} a(x, y)f(y) = g(x)$ where a, f and g take (finite) nonnegative values, which correspond in (1), to kernels b with values in $\mathbb{R} \cup \{+\infty\}$). This approach was systematically developed by Zimmermann [Zim76, Chapter 3], who considered several algebraic structures and allowed in particular the kernel b to take the value $-\infty$. The method of Vorobyev and Zimmermann is one of the basic tools in max-plus linear algebra, and it has been instrumental in the understanding of the geometry of images of max-plus linear operators. It has been the source of several important developments, including the characterisation by Butković [But94, But00] of locally injective (“strongly regular”) max-plus linear maps in terms of optimal assignment problems.

We extend here Vorobyev’s theorem to the case of functional Galois connections. We use an adapted notion of subdifferentials, which is similar to that introduced by Martínez-Legaz and Singer [MLS95]. In the special case of Moreau conjugacies, subdifferentials were introduced by Balder [Bal77], Dolecki and Kurcyusz [DK78], and Lindberg [Lin79] (see also [ML88]). We show that the existence (Section 3) and uniqueness (Section 4) of the solution of (\mathcal{P}) are characterised in terms of coverings and minimal coverings by sets which are inverses of subdifferentials of g . As in the work of Zimmermann, we obtain an algorithm to check the existence and uniqueness of the solution of (\mathcal{P}) , when X and Y are finite (Section 5). We also illustrate our results on various Moreau conjugacies (Section 6). When B is the Legendre-Fenchel transform, our results show (see Section 6.1) that essentially smooth convex functions have a unique preimage by the Legendre-Fenchel transform, a fact which is the essence of the classical Gärtner-Ellis theorem, see e.g. [DZ93, Theorem 2.3.6.(c)] for a general presentation (the use of the uniqueness of the preimage of the Legendre-Fenchel transform was made explicit by O’Brien and Vervaat [OV95, Theorem 4.1 (c)], Gulinsky [Gul03, Theorems 4.7 and 5.3] and Puhalskii [Puh94, Lemmas 3.2 and 3.5]). In Section 6.3, we consider the Moreau conjugacy with kernel $b(x, y) = -\omega(x - y)$, where ω is a nonnegative, continuous, and subadditive map. Spaces of Lipschitz or Hölder continuous maps arise as images of such conjugacies. In Section 6.4, we consider the Moreau conjugacy with kernel $b(x, y) = -x''\|x' - y\|^p$, where $x = (x', x'')$, $x'' \geq 0$ and $p > 0$. This conjugacy was already studied by Dolecki and Kurcyusz [DK78].

Problem (\mathcal{P}) arises when looking for the rate function in large deviations: further applications of our results are given in [AGK04], where we use our characterisations of the uniqueness in Problem (\mathcal{P}) to give a new proof, as well as

generalisations, of the Gärtner-Ellis theorem. A second motivation arises from optimal control: reconstructing the initial condition of an Hamilton-Jacobi equation from the final value is a special case of Problem (\mathcal{P}) , which has been studied, under convexity assumptions, by Goebel and Rockafellar [GR02]. Finally, Problem (\mathcal{P}) arises in the characterisation of dual solutions of the Monge-Kantorovitch mass transfer problem (see [RR98, Vil03] for general presentations). Subdifferentials associated to Moreau conjugacies are instrumental in this theory, as shown by Rüschendorf [Rüs91, Rüs95] (see also [RR98, Section 3.3]).

The results of the present paper were announced in [AGK02].

Acknowledgement. We thank Ivan Singer and an anonymous referee for their helpful comments on a preliminary version of this paper. In particular, Ivan Singer pointed out previous works on generalised subdifferentials, and the anonymous referee pointed out the seminal work of Vorobyev.

2. Representation of Functional Galois Connections

Many basic results of convex analysis are specialisations of general properties of Galois connections in lattices, that we next recall. Galois connections between lattices of subsets were introduced by Birkhoff, in the initial edition of [Bir95]. Galois connections between general lattices were introduced by Ore [Ore44]. (The word Galois correspondence is sometimes used as a synonym of Galois connection.) The proofs of the following results can be found in [Bir95, Chapter V, Section 8], [DJLC53], [BJ72, chapter 1, Section 2], [BCOQ92, Section 4.4.2], and [GHK⁺80, Chapter 0, Section 3].

Let $(\mathcal{F}, \leq_{\mathcal{F}})$ and $(\mathcal{G}, \leq_{\mathcal{G}})$ be two partially ordered sets, and let $B : \mathcal{F} \rightarrow \mathcal{G}$ and $C : \mathcal{G} \rightarrow \mathcal{F}$. We say that B is *antitone* if $f \leq_{\mathcal{F}} f' \implies Bf' \leq_{\mathcal{G}} Bf$. The pair (B, C) is a *dual Galois connection* between \mathcal{F} and \mathcal{G} if it satisfies one of the following equivalent conditions:

- (2a) $I_{\mathcal{F}} \geq_{\mathcal{F}} CB, \quad I_{\mathcal{G}} \geq_{\mathcal{G}} BC, \quad \text{and } B, C \text{ are antitone maps,}$
- (2b) $(g \geq_{\mathcal{G}} Bf \iff f \geq_{\mathcal{F}} Cg) \quad \forall f \in \mathcal{F}, g \in \mathcal{G},$
- (2c) $Cg = \min_{\mathcal{F}} \{f \mid g \geq_{\mathcal{G}} Bf\} \quad \forall g \in \mathcal{G},$
- (2d) $Bf = \min_{\mathcal{G}} \{g \mid f \geq_{\mathcal{F}} Cg\} \quad \forall f \in \mathcal{F},$

where $\min_{\mathcal{F}}$ (resp. $\min_{\mathcal{G}}$) denotes the minimum element of a set for the order $\leq_{\mathcal{F}}$ (resp. $\leq_{\mathcal{G}}$), and where $I_{\mathcal{A}}$ denotes the identity on a set \mathcal{A} .

It follows from (2c) that for any B , there is at most one map C such that (B, C) is a dual Galois connection. We denote this C by B° . It also follows from (2c) that for all $g \in \mathcal{G}$ and $h \in \mathcal{F}$,

$$(3) \quad g = Bh \Rightarrow B^\circ g = \min_{\mathcal{F}} \{f \mid g = Bf\}.$$

In particular,

$$(4) \quad B^\circ = B^{-1} \quad \text{if } B \text{ is invertible.}$$

The two inequalities in (2a) imply that

$$BB^\circ B = B \quad \text{and} \quad B^\circ BB^\circ = B^\circ.$$

From this, or from (3), one deduces

$$Bf = g \text{ has a solution } f \in \mathcal{F} \iff BB^\circ g = g.$$

If (B, C) is a dual Galois connection, so does (C, B) , by symmetry. Hence, $(B^\circ)^\circ = B$. If B yields a dual Galois connection, then

$$(5) \quad B(\inf_{\mathcal{F}} F) = \sup_{\mathcal{G}} \{Bf \mid f \in F\},$$

for any subset F of \mathcal{F} such that the infimum of F exists. In particular, if \mathcal{F} has a maximum element, $\top_{\mathcal{F}}$, we get by specialising (5) to $F = \emptyset$ that \mathcal{G} has a minimum element, $\perp_{\mathcal{G}}$, and

$$(6) \quad B(\top_{\mathcal{F}}) = \perp_{\mathcal{G}}.$$

Moreover, if \mathcal{F} is a complete ordered set, i.e., if any subset of \mathcal{F} has a greatest lower bound, property (5) characterises the maps B that yield a dual Galois connection.

Ordinary (non dual) *Galois connections* are defined by reversing the order relation of \mathcal{F} and \mathcal{G} in (2). One also finds in the literature the names of *residuated maps* B and *dually residuated maps* C , which are defined by reversing the order of \mathcal{F} , but not the order of \mathcal{G} , in (2) (see for instance [BJ72, BCOQ92]). All these notions are equivalent.

We call *lattice of functions* a sublattice \mathcal{F} of S^Y , where (S, \leq) is a lattice, Y is a set, and S^Y is equipped with the product ordering (that we still denote by \leq). When $\mathcal{F} \subset S^Y$ and $\mathcal{G} \subset T^X$ are lattices of functions, we say that (B, B°) is a (dual) *functional Galois connection*.

When S has a maximum element \top_S , $y \in Y$ and $s \in S$, we denote by δ_y^s the map:

$$\delta_y^s \in S^Y : \quad \delta_y^s(y') = \begin{cases} s & \text{if } y' = y, \\ \top_S & \text{otherwise,} \end{cases}$$

that we call the *Dirac function* at point $y \in Y$ with value $s \in S$.

THEOREM 2.1. *Let S, T be two lattices that have a maximum element, let X, Y be arbitrary nonempty sets and let $\mathcal{F} \subset S^Y$ (resp. $\mathcal{G} \subset T^X$) be a lattice of functions containing all the Dirac functions of S^Y (resp. T^X). Then (B, B°) is a dual Galois connection between \mathcal{F} and \mathcal{G} if, and only if, there exist two maps $b : X \times Y \times S \rightarrow T$ and $b^\circ : Y \times X \times T \rightarrow S$ such that: for all $(x, y) \in X \times Y$, $(b(x, y, \cdot), b^\circ(y, x, \cdot))$ is a dual Galois connection between S and T ; for all $(x, t) \in X \times T$, $b^\circ(\cdot, x, t) \in \mathcal{F}$; for all $(y, s) \in Y \times S$, $b(\cdot, y, s) \in \mathcal{G}$; and*

$$(7a) \quad Bf = \sup_{\mathcal{G}} \{b(\cdot, y, f(y)) \mid y \in Y\}, \quad \forall f \in \mathcal{F},$$

$$(7b) \quad B^\circ g = \sup_{\mathcal{F}} \{b^\circ(\cdot, x, g(x)) \mid x \in X\}, \quad \forall g \in \mathcal{G}.$$

In this case, the maps b and b° are uniquely determined by (B, B°) , since $b(\cdot, y, s) = B\delta_y^s$ and $b^\circ(\cdot, x, t) = B^\circ\delta_x^t$ for all $s \in S, t \in T, x \in X$ and $y \in Y$.

Theorem 2.1 was inspired by a “Riesz representation theorem” of Maslov and Kolokoltsov [KM87, Kol90, Kol92] (see also [KM97, Theorem 1.4]) which is similar to Theorem 2.1: it applies to a continuous map B between (non-complete) lattices of *continuous* functions \mathcal{F} and \mathcal{G} , with $S = T = \overline{\mathbb{R}}$, assuming that B preserves finite sups. Martínez-Legaz and Singer obtained in [MLS90, Theorems 3.1 and 3.5] (see also [Sin97, Theorem 7.3]) the same conclusions as in Theorem 2.1, in the special case where $\mathcal{F} = S^Y$, $\mathcal{G} = T^X$ and S and T are complete lattices (Theorem 7.3 of [Sin97] is stated in the case where S and T are included in $\overline{\mathbb{R}}$, but it is remarked in [Sin97, page 419] that this result is valid for general complete lattices S and T).

Theorem 2.1 allows us to consider the case where $\mathcal{F} = \text{lsc}(Y, S)$ is the set of *lower semicontinuous*, or *l.s.c.*, maps from Y to S . Here, we say that a map $f : Y \rightarrow S$ is l.s.c. if for all $s \in S$, the sublevel set $\{y \in Y \mid f(y) \leq s\}$ is closed. When Y is a T_1 topological space and S has a maximum element, the Dirac functions are l.s.c., so that Theorem 2.1 can be applied. In this case, $\sup_{\mathcal{F}} = \sup$ since the sup of l.s.c. maps is l.s.c., and Theorem 2.1 shows that $b^\circ(\cdot, x, t)$ is l.s.c. We shall see in Proposition 2.3 below that $b(x, \cdot, s)$ is also l.s.c.

REMARK 2.2. If $S, T, \mathcal{F}, \mathcal{G}$ are as in Theorem 2.1, \mathcal{F} has a maximum element, namely, the constant function $y \mapsto \top_S$ (which necessarily belongs to \mathcal{F} because it is equal to the Dirac function $\delta_y^{\top_S}$ for any $y \in Y$). Then, if (B, B°) is a dual Galois connection between \mathcal{F} and \mathcal{G} , the remark before (6) shows that \mathcal{G} has a minimum element. Moreover, by Theorem 2.1, the existence of a dual Galois connection between \mathcal{F} and \mathcal{G} implies the existence of dual Galois connection between S and T , hence, by (6), T has a minimum element, \perp_T . Since \mathcal{G} contains all the Dirac functions, the minimum element of \mathcal{G} , $\perp_{\mathcal{G}}$, is such that $\perp_{\mathcal{G}}(x) \leq \delta_x^{\perp_T}(x) = \perp_T$ for all $x \in X$, hence $\perp_{\mathcal{G}}$ is necessarily the constant function $x \mapsto \perp_T$. Symmetrically, S (resp. \mathcal{F}) has a minimum element, \perp_S (resp. the constant function $y \mapsto \perp_S$).

PROOF OF THEOREM 2.1. The proof of Theorem 2.1 is similar to that of Theorems 3.1 and 3.5 in [MLS90]. We give it for completeness. Let us first assume that (B, B°) is a dual Galois connection, and define

$$b(\cdot, y, s) = B\delta_y^s \in \mathcal{G}, \quad b^\circ(\cdot, x, t) = B^\circ\delta_x^t \in \mathcal{F}.$$

We have

$$\begin{aligned} b(x, y, s) \leq t &\iff B\delta_y^s(x) \leq t \\ &\iff B\delta_y^s \leq \delta_x^t \\ &\iff B^\circ\delta_x^t \leq \delta_y^s \quad (\text{by (2b)}) \\ &\iff b^\circ(y, x, t) \leq s, \end{aligned}$$

which shows, by (2b), that $(b(x, y, \cdot), b^\circ(y, x, \cdot))$ is a dual Galois connection between S and T .

Using (5) and $f = \inf_{\mathcal{F}}\{\delta_y^{f(y)} \mid y \in Y\}$, which holds for all $f \in \mathcal{F}$, we get (7a). The representation (7b) is obtained by symmetry.

Conversely, let us assume that (B, B°) are defined by (7) where b and b° satisfy the conditions of the theorem. (This means in particular that the $\sup_{\mathcal{G}}$ and $\sup_{\mathcal{F}}$ in (7) exist.) Then, applying (7a) to $\delta_y^s \in \mathcal{F}$ and using (6), we get that $b(\cdot, y, s) = B\delta_y^s$. Similarly, $b^\circ(\cdot, x, t) = B^\circ\delta_x^t$. Moreover, for all $f \in \mathcal{F}, g \in \mathcal{G}$,

$$\begin{aligned} Bf \leq g &\iff b(\cdot, y, f(y)) \leq g, \forall y \in Y \\ &\iff b(x, y, f(y)) \leq g(x), \forall (x, y) \in X \times Y \\ &\quad (\text{since } \leq \text{ is the product ordering on } \mathcal{G}) \\ &\iff b^\circ(y, x, g(x)) \leq f(y), \forall (x, y) \in X \times Y \quad (\text{by (2b)}) \\ &\iff B^\circ g \leq f, \end{aligned}$$

which, by (2b) again, shows that (B, B°) is a dual Galois connection. \square

PROPOSITION 2.3. *If (b, b°) are as in Theorem 2.1, where Y is a T_1 topological space and $\mathcal{F} = \text{lsc}(Y, S)$, then, for all $(x, s) \in X \times S$, the map $b(x, \cdot, s) : Y \rightarrow T$ is l.s.c.*

PROOF. By Theorem 2.1, $b^\circ(\cdot, x, t) \in \mathcal{F} = \text{lsc}(Y, S)$, for all $(x, t) \in X \times T$. The equivalence (2b) shows that $b^\circ(\cdot, x, t)$ is l.s.c. for all $t \in T$, if, and only if, $b(x, \cdot, s)$ is l.s.c. for all $s \in S$. \square

By symmetry, when X is a T_1 topological space, and $\mathcal{G} = \text{lsc}(X, T)$, the map $b^\circ(y, \cdot, t)$ is l.s.c. for all $(y, t) \in Y \times T$.

We say that \mathcal{F} is a *lattice of subsets* if there exists a set Y such that $\mathcal{F} \subset \mathcal{P}(Y)$, the set of all subsets of Y , and \mathcal{F} is a lattice for the \subset ordering. Taking for S the complete lattice of Booleans $(\{0, 1\}, \leq)$, we can identify \mathcal{F} to a lattice of functions included in S^Y , by using the lattice isomorphism: $F \in \mathcal{P}(Y) \mapsto 1_F$, where $1_F(y) = 1$ if $y \in F$ and $1_F(y) = 0$ otherwise. Hence, specialising Theorem 2.1 to the case where S and T are equal to the lattice of Booleans, and considering non dual Galois connections, we get:

COROLLARY 2.4. *Let X, Y be arbitrary nonempty sets and let $\mathcal{F} \subset \mathcal{P}(Y)$ (resp. $\mathcal{G} \subset \mathcal{P}(X)$) be a lattice of subsets containing the empty set and all singletons of Y (resp. X). Then (B, B°) is a Galois connection between \mathcal{F} and \mathcal{G} if, and only if, there exists a set $\mathcal{B} \subset X \times Y$ such that: for all $x \in X$, $\mathcal{B}_x = \{y \in Y \mid (x, y) \in \mathcal{B}\} \in \mathcal{F}$; for all $y \in Y$, $\mathcal{B}^y = \{x \in X \mid (x, y) \in \mathcal{B}\} \in \mathcal{G}$; and*

$$(8a) \quad BF = \inf_{\mathcal{G}} \{\mathcal{B}^y \mid y \in F\}, \quad \forall F \in \mathcal{F},$$

$$(8b) \quad B^\circ G = \inf_{\mathcal{F}} \{\mathcal{B}_x \mid x \in G\}, \quad \forall G \in \mathcal{G}.$$

In this case, the set \mathcal{B} is uniquely determined by (B, B°) , since $\mathcal{B} = \bigcup_{y \in Y} B(\{y\}) \times \{y\} = \bigcup_{x \in X} \{x\} \times B^\circ(\{x\})$.

When \mathcal{F} (resp. \mathcal{G}) is stable by taking arbitrary intersections, $\inf_{\mathcal{F}}$ (resp. $\inf_{\mathcal{G}}$) coincides with the intersection operation.

The conclusions of Corollary 2.4 were obtained by Everett, when $\mathcal{F} \subset \mathcal{P}(Y)$ and $\mathcal{G} \subset \mathcal{P}(X)$ are complete distributive lattices (see Theorem 5 of [Eve44], the remark following its proof, and Section 8 of Chapter 5, page 124 of [Bir95]). In the special case where $\mathcal{F} = \mathcal{P}(Y)$ and $\mathcal{G} = \mathcal{P}(X)$, the conclusions of Corollary 2.4 were obtained in [Sin86, Theorem 1.1] and [MLS90, Theorem 3.3 and Remark 3.2] using ideas of abstract convex analysis.

REMARK 2.5. Many classical Galois connections are of the form (8) (but \mathcal{F} and \mathcal{G} need not contain the singletons and the empty set). For instance, if Y is an extension of a field K , if \mathcal{F} is the set of intermediate fields $F: K \subset F \subset Y$, if X is the group of automorphisms of Y fixing every element of K , and if \mathcal{G} is the set of subgroups of X , we obtain the original Galois correspondence by setting $\mathcal{B} = \{(g, y) \in X \times Y \mid g(y) = y\}$.

In the sequel, we shall only consider the case where $S = T = \overline{\mathbb{R}}$. In this case, the property that $(b(x, y, \cdot), b^\circ(y, x, \cdot))$ is a dual Galois connection can be made explicit:

LEMMA 2.6. *A map $h: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ yields a dual Galois connection if, and only if, h is nonincreasing, right-continuous, and $h(+\infty) = -\infty$.*

PROOF. This follows readily from the characterisation (5) of dual Galois connections between complete lattices. \square

Since h is nonincreasing, one can replace right-continuous by l.s.c. in Lemma 2.6 as is done in the statement of [MLS90, Theorem 3.2].

EXAMPLE 2.7. When $b \in \overline{\mathbb{R}}$, the map $h : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$, $\lambda \mapsto b - \lambda$, with the convention that $-\infty$ is absorbing for addition, yields a dual Galois connection (by Lemma 2.6). Moreover $h^\circ = h$.

EXAMPLE 2.8. Let $S = T = \overline{\mathbb{R}}$, and $X, Y, \mathcal{F}, \mathcal{G}$ be as in Theorem 2.1. Assume in addition that \mathcal{F} and \mathcal{G} are stable by the addition of a constant, again with the convention that $-\infty$ is absorbing for addition. Let $\bar{b} : X \times Y \rightarrow \overline{\mathbb{R}}$ be a map, and let $B : \mathcal{F} \rightarrow \mathcal{G}$ and $B^\circ : \mathcal{G} \rightarrow \mathcal{F}$ be defined by (7) with

$$(9) \quad b(x, y, \alpha) = b^\circ(y, x, \alpha) = \bar{b}(x, y) - \alpha \quad \forall x \in X, y \in Y, \alpha \in \overline{\mathbb{R}}.$$

Theorem 2.1 shows that (B, B°) is a dual Galois connection if, and only if, $\bar{b}(x, \cdot) \in \mathcal{F}$ for all $x \in X$, and $\bar{b}(\cdot, y) \in \mathcal{G}$ for all $y \in Y$. This result can be applied, in particular, when $\mathcal{F} = \text{lsc}(Y, \overline{\mathbb{R}})$ and $\mathcal{G} = \overline{\mathbb{R}}^X$. Such functional Galois connections are called *Moreau conjugacies* [Mor70] and have been considered by several authors (see for instance [DK78] and [MLS90, Section 5]). In particular, taking two topological vector spaces in duality X and Y , and $\bar{b} : X \times Y \rightarrow \overline{\mathbb{R}}$, $(x, y) \mapsto \langle x, y \rangle$, we obtain the classical Legendre-Fenchel transform $f \mapsto Bf = f^*$.

REMARK 2.9. The set $\overline{\mathbb{R}}$ can be equipped with the semiring structure of $\overline{\mathbb{R}}_{\max}$, in which the addition is $(a, b) \mapsto \max(a, b)$ and the multiplication is $(a, b) \mapsto a + b$, with the convention that $-\infty$ is absorbing for the multiplication of this semiring. Then, if Z is a set, $\overline{\mathbb{R}}^Z$ can be equipped with two different $\overline{\mathbb{R}}_{\max}$ -semimodule structures. The *natural semimodule*, denoted $\overline{\mathbb{R}}_{\max}^Z$, is obtained by taking the addition $(f, f') \mapsto f \oplus f'$, with $(f \oplus f')(z) = \max(f(z), f'(z))$ for all $z \in Z$, and the action $(\lambda, f) \mapsto \lambda \cdot f$ with $(\lambda \cdot f)(z) = \lambda + f(z)$ for all $z \in Z$, again with the convention that $-\infty$ is absorbing. The *opposite semimodule*, denoted $(\overline{\mathbb{R}}_{\max}^Z)^{\text{op}}$, is obtained by taking the addition $(f, f') \mapsto f \oplus' f'$, with $(f \oplus' f')(z) = \min(f(z), f'(z))$ for all $z \in Z$, and the action $(\lambda, f) \mapsto \lambda \cdot' f$ with $(\lambda \cdot' f)(z) = -\lambda + f(z)$ for all $z \in Z$, with the dual convention that $+\infty$ is absorbing (see [CGQ04]). Then the Moreau conjugacies, i.e., the functional Galois connections of Example 2.8, are $\overline{\mathbb{R}}_{\max}$ -linear from $(\overline{\mathbb{R}}_{\max}^Z)^{\text{op}}$ to $\overline{\mathbb{R}}_{\max}^Z$.

3. Existence of Solutions of $Bf = g$

3.1. Statement of the Existence Result. In the following, we take $S = T = \overline{\mathbb{R}}$, we assume that X and Y are Hausdorff topological spaces, and take $\mathcal{F} = \text{lsc}(Y, \overline{\mathbb{R}})$, $\mathcal{G} = \overline{\mathbb{R}}^X$, together with B, B°, b, b° as in Theorem 2.1. This includes the case where $\mathcal{F} = \overline{\mathbb{R}}^Y$ and X, Y are arbitrary sets, which is obtained by taking discrete topologies on X and Y . Since, by Theorem 2.1, $(b(x, y, \cdot), b^\circ(y, x, \cdot))$ is a dual Galois connection, Lemma 2.6 shows that $b(x, y, \cdot)$ and $b^\circ(y, x, \cdot)$ are non-increasing and right-continuous maps from $\overline{\mathbb{R}}$ to itself, which take the value $-\infty$ at $+\infty$.

We shall assume in the sequel that there is a subset $\mathcal{S} \subset X \times Y$ satisfying:

- (A1) $\mathcal{S}_x = \{y \in Y \mid (x, y) \in \mathcal{S}\} \neq \emptyset$, for all $x \in X$;
- (A2) $\mathcal{S}^y = \{x \in X \mid (x, y) \in \mathcal{S}\} \neq \emptyset$, for all $y \in Y$;
- (A3) $b(x, y, \cdot)$ is a bijection $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ for all $(x, y) \in \mathcal{S}$;
- (A4) $b(x, y, \cdot) \equiv -\infty$, for $(x, y) \in X \times Y \setminus \mathcal{S}$.

When B is the Moreau conjugacy given by (7,9), Assumptions (A1–A4) are satisfied if, and only if, $\bar{b}(x, y) \in \mathbb{R} \cup \{-\infty\}$ for all $(x, y) \in X \times Y$, and $\mathcal{S} := \{(x, y) \in X \times Y \mid \bar{b}(x, y) \in \mathbb{R}\}$ satisfies (A1–A2), that is, for all $x \in X$ and

$y \in Y$, $\bar{b}(x, \cdot)$ and $\bar{b}(\cdot, y)$ are not identically $-\infty$. These assumptions are fulfilled in particular for the kernel of the Legendre-Fenchel transform.

Rather than Problem (\mathcal{P}) , we will consider the more general problem:

$$(\mathcal{P}') : \quad \text{Find } f \in \mathcal{F} \text{ such that } Bf \leq g \text{ and } Bf(x) = g(x) \text{ for all } x \in X',$$

where $g \in \mathcal{G}$ and $X' \subset X$ are given.

To state our results, we need some definitions and notations. When B is the Legendre-Fenchel transform, these notations and definitions will correspond to those defined classically in convex analysis. First, for any map g from a topological space Z to $\overline{\mathbb{R}}$, we define the *lower domain*, *upper domain*, *domain*, and *inner domain*:

$$\begin{aligned} \text{ldom}(g) &= \{z \in Z \mid g(z) < +\infty\}, \\ \text{udom}(g) &= \{z \in Z \mid g(z) > -\infty\}, \\ \text{dom}(g) &= \text{ldom}(g) \cap \text{udom}(g), \\ \text{idom}(g) &= \{z \in \text{dom}(g) \mid \limsup_{z' \rightarrow z} g(z') < +\infty\}. \end{aligned}$$

The set $\text{idom}(g) \subset \text{dom}(g)$ is an open subset of $\text{udom}(g)$. When Z is endowed with the discrete topology, $\text{idom}(g) = \text{dom}(g)$. We say that g is *proper* if $g(z) \neq -\infty$ for all $z \in Z$ and if there exists $z \in Z$ such that $g(z) \neq +\infty$, which means that $\text{udom}(g) = Z$ and $\text{dom}(g) \neq \emptyset$.

We shall use the following variant of the general notion of subdifferentials of dualities introduced by Martínez-Legaz and Singer [MLS95] (see Remark 3.1 for a comparison). Given $f \in \mathcal{F}$ and $y \in Y$, we call the *subdifferential* of f at y with respect to b (or B), and we denote by $\partial^b f(y)$, or $\partial f(y)$ for brevity, the set:

$$(10a) \quad \partial f(y) = \{x \in X \mid (x, y) \in \mathcal{S}, \quad b(x, y', f(y')) \leq b(x, y, f(y)) \quad \forall y' \in Y\}.$$

For $g \in \mathcal{G}$, and $x \in X$, the subdifferential of g at x with respect to b° , $\partial^{b^\circ} g(x)$, that we denote $\partial^o g(x)$ for brevity, is given by:

$$(10b) \quad \partial^o g(x) = \{y \in Y \mid (x, y) \in \mathcal{S}, \quad b^\circ(y, x', g(x')) \leq b^\circ(y, x, g(x)) \quad \forall x' \in X\}.$$

Then

$$(11a) \quad \partial f(y) = \{x \in X \mid (x, y) \in \mathcal{S} \text{ and } Bf(x) = b(x, y, f(y))\},$$

$$(11b) \quad \partial^o g(x) = \{y \in Y \mid (x, y) \in \mathcal{S} \text{ and } B^\circ g(y) = b^\circ(y, x, g(x))\},$$

and when $b(x, y, \alpha) = \langle x, y \rangle - \alpha$ is the kernel of the Legendre-Fenchel transform, we recover the classical definition of subdifferentials.

REMARK 3.1. From (A3) and (4), $b(x, y, \cdot)$ is a bijection with inverse $b^\circ(y, x, \cdot)$, for all $(x, y) \in \mathcal{S}$. Hence,

$$\partial f(y) = \{x \in X \mid (x, y) \in \mathcal{S} \text{ and } b^\circ(y, x, Bf(x)) = f(y)\},$$

$$\partial^o g(x) = \{y \in Y \mid (x, y) \in \mathcal{S} \text{ and } b(x, y, B^\circ g(y)) = g(x)\}.$$

In [MLS95], the subdifferential $\partial^B f(y)$ of f at y with respect to B is defined by $\partial^B f(y) = \{x \in X \mid b^\circ(y, x, Bf(x)) = f(y)\}$. Hence, under Assumptions (A1–A4), the subdifferential $\partial f(y)$ and the subdifferential $\partial^B f(y)$ of Martínez-Legaz and Singer differ only when $f(y) = -\infty$. In this case $\partial^B f(y) = X$ and $\partial f(y) = \mathcal{S}^y$. Hence, $\partial f(y) \subset \partial^B f(y)$ holds for all $y \in Y$. Moreover $\partial^B f(y) = \emptyset$ if, and only

if, $\partial f(y) = \emptyset$. With these remarks in mind, one may rephrase equivalently all the subsequent results in terms of ∂^B and ∂^{B° .

As was observed in [MLS95], the generalised subdifferentials have a geometric interpretation. Let us fix $(x, y) \in \mathcal{S}$. Thanks to Assumption (A3), there is a unique function of the family $\{b(\cdot, y, \lambda)\}_{\lambda \in \mathbb{R}}$ which takes the value $g(x)$ at point x : this function is obtained for $\lambda = b^\circ(y, x, g(x))$, since, as observed in Remark 3.1, $b^\circ(y, x, \cdot)$ is the inverse of $b(x, y, \cdot)$. Let us call this function the *curve of direction y meeting g at point x* . By (2b), $y \in \partial^B g(x)$ if, and only if, $g(x') \geq b(x', y, b^\circ(y, x, g(x)))$, for all $x' \in X$, which means that *the curve of direction y meeting g at x is below g* . When $B = B^\circ$ is the Legendre-Fenchel transform, the curves become lines, and we recover the classical interpretation of subdifferentials.

EXAMPLE 3.2. The geometrical interpretation of subdifferentials is illustrated in Figure 1, where $X = Y = \mathbb{R}$, b is given by (9) with $\bar{b}(x, y) = -|x - y|$, $g(x) = x^2/2$ for $x \leq 0$, $g(x) = x$ for $x \in [0, 1]$, $g(x) = 1$ for $x \in [1, 3]$, and $g(x) = x/3 - 1$ for $x \in [3, \infty)$. We have $\partial^B g(x) = \emptyset$ for $x \in (-\infty, -1) \cup (2, 3)$, $\partial^B g(x) = \{x\}$ for $x \in (-1, 0) \cup [1, 2] \cup (3, \infty)$, $\partial^B g(x) = [x, 1]$ for $x \in [0, 1]$, $\partial^B g(-1) = (-\infty, -1]$ and $\partial^B g(3) = [2, 3]$. This can be checked by looking at the curves meeting g at points $-1, 1/2, 3$ and 4 , which are depicted on the figure.

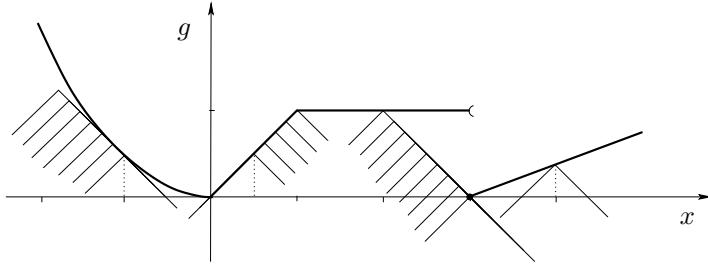


FIGURE 1. Geometric interpretation of subdifferentials

DEFINITION 3.3. When F is a map from a set Z to the set $\mathcal{P}(W)$ of all subsets of some set W , we denote by F^{-1} the map from W to $\mathcal{P}(Z)$ given by $F^{-1}(w) = \{z \in Z \mid w \in F(z)\}$, and we define the domain of F : $\text{dom}(F) := \{z \in Z \mid F(z) \neq \emptyset\} = \cup_{w \in W} F^{-1}(w)$. If $Z' \subset Z$ and $W' \subset W$, we say that $\{F(z)\}_{z \in Z'}$ is a *covering* of W' if $\cup_{z \in Z'} F(z) \supset W'$.

When F, Z, Z', W are as in Definition 3.3, the family $\{F^{-1}(w)\}_{w \in W}$ is a covering of Z' if, and only if, $Z' \subset \text{dom}(F)$. By (10),

$$(12a) \quad (\partial f)^{-1}(x) = \arg \max_{y \in \mathcal{S}_x} b(x, y, f(y)) ,$$

$$(12b) \quad (\partial^B g)^{-1}(y) = \arg \max_{x \in \mathcal{S}^y} b^\circ(y, x, g(x)) .$$

DEFINITION 3.4. We say that b is *continuous in the second variable* if for all $x \in X$ and $\alpha \in \mathbb{R}$, $b(x, \cdot, \alpha)$ is continuous. We say that b is *coercive* if for all $x \in X$, all neighbourhoods V of x in X , and all $\alpha \in \mathbb{R}$, the function

$$(13) \quad y \in Y \mapsto b_{x, V}^\alpha(y) = \sup_{z \in V} b(z, y, b^\circ(y, x, \alpha)) ,$$

has relatively compact finite sublevel sets, which means that $\{y \in Y \mid b_{x,V}^\alpha(y) \leq \beta\}$ is relatively compact for all $\beta \in \mathbb{R}$.

The continuity of b in the second variable holds readily when Y is discrete (and in particular when Y is finite). The coercivity of b holds trivially, and independently of the topology on X , when Y is compact (and in particular when Y is finite). If $X = Y = \mathbb{R}^n$ and $b(x, y, \alpha) = \langle x, y \rangle - \alpha$ then b is continuous in the second variable, and for all neighbourhoods V of x , and all $\alpha \in \mathbb{R}$, $b_{x,V}^\alpha(y) \geq \varepsilon \|y\| + \alpha$, for some $\varepsilon > 0$, so that b is coercive. Similarly, if $b(x, y, \alpha) = a\|x - y\|^2 - \alpha$, where $a \in \mathbb{R} \setminus \{0\}$ and $\|\cdot\|$ is the Euclidean norm, then b is continuous in the second variable, and for all neighbourhoods V of x , and all $\alpha \in \mathbb{R}$, $b_{x,V}^\alpha(y) \geq \varepsilon \|y - x\| - 1 + \alpha$, for some $\varepsilon > 0$, so that b is coercive.

We also denote by \mathcal{F}_c the set of all $f \in \mathcal{F}$ such that for all $x \in X$, $y \mapsto b(x, y, f(y))$ has relatively compact finite superlevel sets, which means that for all $\beta \in \mathbb{R}$, the set $\{y \in Y \mid b(x, y, f(y)) \geq \beta\}$ is relatively compact. When Y is compact, $\mathcal{F}_c = \mathcal{F}$.

We shall occasionally make the following assumptions:

- (A5) Y is discrete;
- (A5)' b is continuous in the second variable, and $B^\circ g(y) > -\infty$ for all $y \in Y$;
- (A6) $B^\circ g \in \mathcal{F}_c$;
- (A6)' b is coercive and $X' \subset \text{idom}(g) \cup g^{-1}(-\infty)$.

The assumption that $B^\circ g(y) > -\infty$ for all $y \in Y$ is fulfilled in very general situations: if $g(x) < +\infty$ for all $x \in X$, in particular if $\text{idom}(g) = \text{udom}(g)$ as in Corollary 3.8 below; or if $\mathcal{S} = X \times Y$ and $g \not\equiv +\infty$, which is the case for instance when B is the Legendre-Fenchel transform and g is proper.

The following general existence result is proved in Section 3.2.

THEOREM 3.5. *Let $X' \subset X$, and $g \in \mathcal{G}$. Consider the following statements:*

- (14) *Problem (\mathcal{P}') has a solution,*
- (15) *$X' \subset \text{dom}(\partial^\circ g)$,*
- (16) *$\{(\partial^\circ g)^{-1}(y)\}_{y \in Y}$ is a covering of X' ,*
- (17) *$\{(\partial^\circ g)^{-1}(y)\}_{y \in \text{idom}(B^\circ g)}$ is a covering of $X' \cap \text{udom}(g)$,*
- (18) *$\{(\partial^\circ g)^{-1}(y)\}_{y \in \text{dom}(B^\circ g)}$ is a covering of $X' \cap \text{dom}(g)$.*

We have: (15) \Leftrightarrow (16) \Leftrightarrow (17) \Rightarrow (14, 18). The implication (14) \Rightarrow (17) holds if Assumptions (A5) or (A5)', and (A6) or (A6)', hold. This is the case in particular if Y is finite. The implication (18) \Rightarrow (17) holds when $X' \subset \text{idom}(g) \cup g^{-1}(-\infty)$. Finally, (14–18) are true when $g \equiv +\infty$ or $g \equiv -\infty$.

The most intuitive condition should be (15): it says that $\partial^\circ g(x) \neq \emptyset$, for all $x \in X'$. See Example 3.12 below for an illustration. We stated conditions involving coverings to make it clear that Theorem 3.5 generalises the theorem of Vorob'ev and Zimmermann. Moreover, coverings will be instrumental in the statement of the uniqueness results in Section 4.

REMARK 3.6. As we shall see in Section 3.2, the implication (15) \Rightarrow (14) can be deduced from (21). The implication in (21) was already shown by Martínez-Legaz and Singer [MLS95, Proposition 1.2], using the notion of subdifferential of [MLS95].

We next state some direct corollaries.

COROLLARY 3.7. *Consider $g \in \mathcal{G}$. Assume that Y is finite. Then $Bf = g$ has a solution $f \in \mathcal{F}$ if, and only if, $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in Y}$ is a covering of X . \square*

COROLLARY 3.8. *Consider $g \in \mathcal{G}$ such that $\text{idom}(g) = \text{udom}(g)$. Assume that b is continuous in the second variable and coercive. Then $Bf = g$ has a solution $f \in \mathcal{F}$ if, and only if, $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in Y}$ is a covering of X . \square*

EXAMPLE 3.9. When B is the Legendre-Fenchel transform over \mathbb{R}^n , and g is a l.s.c. proper convex function, Problem (\mathcal{P}) has a solution. A fortiori, Problem $(\mathcal{P})'$ has a solution with $X' = \text{idom}(g)$. Then the implication $(14) \Rightarrow (15)$ of Theorem 3.5 shows that g admits subdifferentials in $\text{idom}(g)$, a well known result since for any l.s.c. convex function g on \mathbb{R}^n , $\text{idom}(g)$ is the interior of $\text{dom}(g)$ (see for instance [Roc70, Theorem 23.4]). Corollary 3.8 shows that if g is finite and locally bounded from above everywhere, then g is l.s.c. and convex if, and only if, it has nonempty subdifferentials everywhere.

The following final corollary is proved in Section 3.2:

COROLLARY 3.10. *Let $g \in \mathcal{G}$. Make Assumptions (A5)', and (A6). Then $Bf = g$ has a solution $f \in \mathcal{F}$ if, and only if, $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in Y}$ is a covering of X . In that case, $g(x) < +\infty$ for all $x \in X$.*

REMARK 3.11. Assumptions (A5) or (A5)', together with (A6) or (A6)' imply that:

$$(A7) \quad \forall x \in X' \cap \text{udom}(g), \exists y \in Y, b(x, y, B^{\circ}g(y)) = \sup_{y' \in Y} b(x, y', B^{\circ}g(y')).$$

Indeed, Assumption (A5) or (A5)' implies that the map $y \mapsto b(x, y, B^{\circ}g(y))$ is u.s.c. for all $x \in X$, whereas Assumption (A6) (resp. (A6)') implies that the map $y \mapsto b(x, y, B^{\circ}g(y))$ has relatively compact finite superlevel sets, for all $x \in X$ (resp. for all $x \in \text{idom}(g) \supset X' \cap \text{udom}(g)$). One can check that the conclusions of Theorem 3.5 remain valid if Assumptions (A5) or (A5)', and (A6) or (A6)', are replaced by the single Assumption (A7). Indeed, the proof of Theorem 3.5 is based on the application of Theorem 3.25 to $f = B^{\circ}g$, and on the inclusion of $X' \cap \text{udom}(g)$ in the set X_0 which appears in (28), and the proof of Theorem 3.25 is precisely based on (28) (the other arguments do not need any assumption). Similarly, the proof of Corollary 3.10 remains valid if all the assumptions are replaced by Assumption (A7) with $X' \cap \text{udom}(g)$ replaced by X .

We shall give examples of application of Theorem 3.5 (and of its corollaries) in Sections 5 and 6 below. We next give examples illuminating the role of the technical assumptions in Theorem 3.5.

EXAMPLE 3.12. We first show a limitation of Theorem 3.5. Consider again the kernel b and the map g of Example 3.2. We claim that:

$$(19) \quad \text{Problem } (\mathcal{P}') \text{ has a solution, if, and only if, } X' \subset [-1, 2] \cup [3, \infty).$$

Indeed, $\text{dom}(\partial^{\circ}g) = \{x \in \mathbb{R} \mid \partial^{\circ}g(x) \neq \emptyset\} = [-1, 2] \cup [3, \infty)$, so that the “if” part of (19) follows from the implication $(15) \Rightarrow (14)$ in Theorem 3.5. However, the “only if” part of (19) cannot be derived from Theorem 3.5, because Assumptions (A6) and (A6)' do not hold (Assumption (A5)' is satisfied). Indeed, we shall see in the more general setting of Section 6.3 that the kernel b is not coercive. Moreover, one has $B^{\circ}g = -BB^{\circ}g$, $BB^{\circ}g(x) = g(x)$ for $x \in [-1, 2] \cup [3, \infty)$,

$BB^\circ g(x) = -x - 1/2$ for $x \in (-\infty, -1]$ and $BB^\circ g(x) = 3 - x$ for $x \in (2, 3)$. Then $b(x, y, B^\circ g(y)) = -x - 1/2$ for $y \leq x$ and $y \leq -1$, and $b(x, y, B^\circ g(y))$ goes to $-\infty$ when y goes to $+\infty$, which implies that $B^\circ g \notin \mathcal{F}_c$. Therefore, we cannot apply the implication (14) \Rightarrow (17) in Theorem 3.5 to characterise the cases where Problem (\mathcal{P}') has a solution. However, if Problem (\mathcal{P}') has a solution, then by Lemma 3.18 below and (2a), $B^\circ g$ is also a solution of (\mathcal{P}') . Therefore, X' must be included in the set $\{x \in X \mid BB^\circ g(x) = g(x)\}$, which is equal to $[-1, 2] \cup [3, \infty)$, by the previous computations. This shows the “only if” part of (19). This conclusion can also be obtained by using the arguments of Remark 3.11 and the fact that Assumption (A7) holds for all subsets X' of \mathbb{R} . Note also that Theorem 3.5 characterises the solvability of Problem (\mathcal{P}') for any map $g \in \mathcal{G}$ such that $\lim_{|x| \rightarrow \infty} g(x) - |x| = -\infty$, since in this case, $B^\circ g \in \mathcal{F}_c$.

EXAMPLE 3.13. The following counter-example shows that the compactness Assumptions (A6) or (A6)' are useful. Consider $X = \mathbb{R}$, $Y = [1, \infty)$, the Moreau conjugacy given by (7,9) with $\bar{b}(x, y) = 0 \vee (-|x| + 1/y)$, where \vee denotes the sup law, and take the identically zero function g . Then $B^\circ g(y) = 1/y$ and $g = BB^\circ g$, which means that Problem (\mathcal{P}') has a solution with $X' = X$. However, $\partial^\circ g(0) = Y$ and $\partial^\circ g(x) = \emptyset$ for all $x \in \mathbb{R} \setminus \{0\}$, which shows that $\text{dom}(\partial^\circ g) = \{0\}$, hence the covering condition (15) or (16) does not hold. Since Assumption (A5)' is satisfied and $\text{dom}(g) = \text{idom}(g) = X$, Theorem 3.5 implies that b is not coercive and $B^\circ g \notin \mathcal{F}_c$.

EXAMPLE 3.14. The following counter-example shows that Assumption (A3) is useful. Consider $Y = \{y_1\}$, $\mathcal{F} = \overline{\mathbb{R}}^Y$, $X = \{x_1, x_2\}$, $\mathcal{G} = \overline{\mathbb{R}}^X$, and the Moreau conjugacy given by (7,9) with $\bar{b}(x_1, y_1) = 0$, $\bar{b}(x_2, y_1) = +\infty$. We have $Bf(x_1) = -f(y_1)$, $Bf(x_2) = +\infty - f(y_1)$, and $B^\circ g(y_1) = \max(-g(x_1), (+\infty) - g(x_2))$, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Take $g(x_1) = 0$ and $g(x_2) = +\infty$. Then $Bf = g$ has a solution, namely, $f(y_1) = 0$. However, taking any $\mathcal{S} \subset X \times Y$ in (12b), we get $(\partial^\circ g)^{-1}(y_1) \subset \arg \max_{x \in \{x_1, x_2\}} \bar{b}(x, y_1) - g(x) = \{x_1\}$ which does not cover $X = \{x_1, x_2\}$. Therefore, the implication (14) \Rightarrow (16) of Theorem 3.5 does not extend to the case of kernels b which take the value $+\infty$, even when Y is finite (these kernels do not satisfy Assumption (A3)).

Take now $g(x_1) = -\infty$ and $g(x_2) = 0$. Then $Bf = g$ has no solution, but taking $\mathcal{S} = X \times Y$ in (12b), we get $(\partial^\circ g)^{-1}(y_1) = \arg \max_{x \in \{x_1, x_2\}} b^\circ(y_1, x, g(x)) = X$. Therefore, the implication (16) \Rightarrow (14) of Theorem 3.5 does not extend to the case of kernels b which take the value $+\infty$. In this case, one should use rather the definition of subdifferentials of Martínez-Legaz and Singer, for which the implication (16) \Rightarrow (14) can be deduced from [MLS95, Proposition 1.2].

3.2. Additional Properties of B , and Proof of Theorem 3.5. In this section, we state several lemmas and prove successively the different assertions of Theorem 3.5. We first show some properties of the kernels b and b° .

By Theorem 2.1, we know that $(b(x, y, \cdot), b^\circ(y, x, \cdot))$ is a *dual* Galois connection between $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}$. The following result, which uses Assumptions (A3) and (A4), shows that $(b(x, y, \cdot), b^\circ(y, x, \cdot))$ is almost a *(non dual)* Galois connection:

LEMMA 3.15. *For all $(x, y) \in \mathcal{S}$, $b(x, y, \cdot)$ is a decreasing bijection $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ with inverse $b^\circ(y, x, \cdot)$. For all $(x, y) \in X \times Y$, we have*

$$(20a) \quad (b(x, y, \beta) \geq \alpha \text{ and } \alpha, \beta > -\infty) \iff (b^\circ(y, x, \alpha) \geq \beta \text{ and } \alpha, \beta > -\infty).$$

Moreover,

$$(20b) \quad (b(x, y, \beta) \geq \alpha \text{ and } \alpha > -\infty) \implies b^\circ(y, x, \alpha) \geq \beta.$$

PROOF. We already observed in Remark 3.1 that by (A3) and (4), $b^\circ(y, x, \cdot)$ is the inverse of $b(x, y, \cdot)$ when $(x, y) \in \mathcal{S}$. Moreover, by Lemma 2.6, $b(x, y, \cdot)$ and $b^\circ(y, x, \cdot)$ are nonincreasing. When the left-hand side of (20b) is satisfied, we have $(x, y) \in \mathcal{S}$, so $b^\circ(y, x, \cdot)$ is the inverse of $b(x, y, \cdot)$, which shows (20b). Together with the symmetric implication, this shows (20a). \square

PROPOSITION 3.16. *If $g \in \mathcal{G}$ and $g = BB^\circ g$, then $(\partial^\circ g)^{-1} = \partial B^\circ g$.*

PROOF. By (11b), $(\partial^\circ g)^{-1}(y) = \{x \in X \mid (x, y) \in \mathcal{S} \text{ and } b^\circ(y, x, g(x)) = B^\circ g(y)\}$. As noted in Lemma 3.15, for $(x, y) \in \mathcal{S}$, $b^\circ(y, x, \cdot)$ is a bijection with inverse $b(x, y, \cdot)$, so that $(\partial^\circ g)^{-1}(y) = \{x \in X \mid (x, y) \in \mathcal{S} \text{ and } g(x) = b(x, y, B^\circ g(y))\}$. Using $BB^\circ g = g$ and (11a), we get $(\partial^\circ g)^{-1}(y) = \partial B^\circ g(y)$. \square

REMARK 3.17. When B is the Legendre-Fenchel transform, a function is in the image of B , or of B° , if, and only if, it is either convex, l.s.c., and proper, or identically $+\infty$, or identically $-\infty$. Then Proposition 3.16 gives the classical inversion property of subdifferentials, $(\partial g)^{-1} = \partial g^*$, which holds for all convex l.s.c. proper functions g (see for instance [Roc70, Theorem 23.5]).

PROOF OF (15) \Rightarrow (14) OF THEOREM 3.5. Let us assume that $\partial^\circ g(x) \neq \emptyset$, for all $x \in X'$, and let us show that $f = B^\circ g$ satisfies (\mathcal{P}') . Since by (2a), $BB^\circ g \leq g$, it is enough to prove that

$$(21) \quad \partial^\circ g(x) \neq \emptyset \implies BB^\circ g(x) \geq g(x).$$

If $y \in \partial^\circ g(x)$, then, by (11b), $b^\circ(y, x, g(x)) = B^\circ g(y)$ and $(x, y) \in \mathcal{S}$, which yields $BB^\circ g(x) \geq b(x, y, B^\circ g(y)) = b(x, y, b^\circ(y, x, g(x))) = g(x)$, by the first assertion of Lemma 3.15, and (21) is shown. \square

To pursue the proof of Theorem 3.5, we state properties of subdifferentials which generalise Proposition 3.16.

LEMMA 3.18. *Consider $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that $Bf \leq g$, and let $E = \{x \in X \mid Bf(x) = g(x)\}$. Then*

$$(22a) \quad BB^\circ g(x) = g(x), \text{ for all } x \in E,$$

and for all $y \in Y$,

$$(22b) \quad \partial f(y) \cap E = \begin{cases} \partial B^\circ g(y) \cap E = (\partial^\circ g)^{-1}(y) & \text{if } f(y) = B^\circ g(y), \\ \emptyset & \text{otherwise.} \end{cases}$$

PROOF. By (2b), $Bf \leq g$ implies $f \geq B^\circ g$. Since B is antitone, applying B to $f \geq B^\circ g$, we get $Bf \leq BB^\circ g$. By (2a), $BB^\circ g \leq g$, hence $Bf \leq BB^\circ g \leq g$, which implies (22a).

Let $x \in \partial f(y) \cap E$. Then, by (11a), $(x, y) \in \mathcal{S}$ and $Bf(x) = b(x, y, f(y))$. Using the definition of E and the first assertion of Lemma 3.15, we get that $f(y) = b^\circ(y, x, g(x))$. Using $f \geq B^\circ g$ and the definition of B° , we obtain $B^\circ g(y) \leq f(y) = b^\circ(y, x, g(x)) \leq B^\circ g(y)$. Thus, $f(y) = B^\circ g(y)$ and, using (11b), $y \in \partial^\circ g(x)$. This shows that $\partial f(y) \cap E \subset (\partial^\circ g)^{-1}(y)$ for all $y \in Y$, and that $\partial f(y) \cap E = \emptyset$ when $f(y) \neq B^\circ g(y)$. To prove the converse inclusion in (22b), let $y \in Y$ such that $f(y) = B^\circ g(y)$. Let $x \in (\partial^\circ g)^{-1}(y)$, then, by (11b), $(x, y) \in \mathcal{S}$ and $f(y) =$

$B^\circ g(y) = b^\circ(y, x, g(x))$. It follows that $g(x) = b(x, y, f(y)) \leq Bf(x) \leq g(x)$. Hence, $x \in E$ and, by (11a), $x \in \partial f(y)$. This shows that $\partial f(y) \cap E = (\partial^\circ g)^{-1}(y)$ when $f(y) = B^\circ g(y)$. Replacing f by $B^\circ g$, we obtain that $\partial B^\circ g(y) \cap \{x \in X \mid BB^\circ g(x) = g(x)\} = (\partial^\circ g)^{-1}(y)$ for all $y \in Y$. Taking the intersection with E , using (22a) and using $(\partial^\circ g)^{-1}(y) = \partial f(y) \cap E \subset E$, we obtain $\partial B^\circ g(y) \cap E = (\partial^\circ g)^{-1}(y) \cap E = (\partial^\circ g)^{-1}(y)$ when $f(y) = B^\circ g(y)$. \square

LEMMA 3.19. *Consider $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that $Bf \leq g$, and let $E = \{x \in X \mid Bf(x) = g(x)\}$. We have*

- (23a) $(\cup_{y \in f^{-1}(+\infty)} \partial f(y)) \cap E = \cup_{y \in (B^\circ g)^{-1}(+\infty)} (\partial^\circ g)^{-1}(y) = g^{-1}(-\infty)$,
- (23b) $(\cup_{y \in \text{dom}(f)} \partial f(y)) \cap E \subset \cup_{y \in \text{dom}(B^\circ g)} (\partial^\circ g)^{-1}(y) \subset \text{dom}(g)$,
- (23c) $(\cup_{y \in f^{-1}(-\infty)} \partial f(y)) \cap E \subset \cup_{y \in (B^\circ g)^{-1}(-\infty)} (\partial^\circ g)^{-1}(y) \subset g^{-1}(+\infty)$.

PROOF. The first inclusion in (23b) and (23c), and the inclusion of the left hand side term of (23a) in the middle term of (23a), follow readily from (22b). If $x \in (\partial^\circ g)^{-1}(y)$ with $y \in Y$, then, by (11b), $(x, y) \in \mathcal{S}$ and $B^\circ g(y) = b^\circ(y, x, g(x))$. Using that $b^\circ(y, x, \cdot)$ is a decreasing bijection $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$, we get the second inclusions in (23b) and (23c), and also the inclusion of the middle term of (23a) in the right hand side term of (23a). This concludes the proof of (23b) and (23c). It remains to prove the inclusion of the right hand side term of (23a) in the left hand side term of (23a). Consider $x \in X$ such that $g(x) = -\infty$. Since $Bf \leq g$, we get that $x \in E$. Moreover, since $\mathcal{S}_x \neq \emptyset$, there is a $y \in Y$ such that $b(x, y, \cdot)$ is bijective. This implies that $f(y) \geq B^\circ g(y) \geq b^\circ(y, x, g(x)) = b^\circ(y, x, -\infty) = +\infty$, hence $f(y) = +\infty$. Moreover, $Bf(x) = g(x) = -\infty = b(x, y, +\infty) = b(x, y, f(y))$, which shows, by (11a), that $x \in \partial f(y)$. It follows that $x \in \partial f(y) \cap E$ with $f(y) = +\infty$, which concludes the proof. \square

PROOF OF (15) \Leftrightarrow (16) \Leftrightarrow (17) \Rightarrow (18) AND OF (18) \Rightarrow (17), IN THEOREM 3.5.
The equivalence (15) \Leftrightarrow (16) holds trivially by the definition of $\text{dom}(\partial^\circ g)$ and of a covering. By (23a), we get $\cup_{y \in (B^\circ g)^{-1}(+\infty)} (\partial^\circ g)^{-1}(y) = g^{-1}(-\infty) = X \setminus \text{udom}(g)$, and since $(B^\circ g)^{-1}(+\infty) = Y \setminus \text{ldom}(B^\circ g)$, we deduce (16) \Leftrightarrow (17). By (23c), $\cup_{y \in (B^\circ g)^{-1}(-\infty)} (\partial^\circ g)^{-1}(y) \subset g^{-1}(+\infty)$, hence we always have (17) \Rightarrow (18). Since $\text{dom}(B^\circ g) \subset \text{ldom}(B^\circ g)$ and $X' \cap \text{udom}(g) = X' \cap \text{dom}(g)$ when $X' \subset \text{idom}(g) \cup g^{-1}(-\infty)$, we have proved the implication (18) \Rightarrow (17) in Theorem 3.5. \square

Conditions (14–18) of Theorem 3.5 are trivial in the following degenerate cases:

PROPOSITION 3.20. *Let $g \in \mathcal{G}$. We have*

- (24a) $g \equiv +\infty \Leftrightarrow B^\circ g \equiv -\infty$,
- (24b) $g \equiv -\infty \Rightarrow B^\circ g \equiv +\infty$.

In both cases, $BB^\circ g = g$ and $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y}$ is a covering of X . Moreover, if $B^\circ g \equiv +\infty$ and $BB^\circ g = g$, then $g \equiv -\infty$.

PROOF. The implication \Rightarrow in (24a) follows from (6). By symmetry, if $f \in \mathcal{F}$ then $f \equiv +\infty$ implies $Bf \equiv -\infty$. Taking $f = B^\circ g$, we get

$$(25) \quad B^\circ g \equiv +\infty \implies BB^\circ g \equiv -\infty,$$

which implies the last assertion of the lemma. If $g \equiv -\infty$, then for all $y \in Y$, taking $x \in \mathcal{S}^y$, we get $B^\circ g(y) \geq b^\circ(y, x, g(x)) = +\infty$, which shows (24b). By symmetry,

if $f \in \mathcal{F}$ then $f \equiv -\infty$ implies $Bf \equiv +\infty$. Applying this property to $f = B^\circ g$, we get that $B^\circ g \equiv -\infty$ implies $BB^\circ g \equiv +\infty$, and since $g \geq BB^\circ g$, $g \equiv +\infty$, which shows the implication \Leftarrow in (24a), together with $BB^\circ g = g$. When $g \equiv -\infty$, combining (24b) and (25), we also get $BB^\circ g = g$. Moreover, since $\text{udom}(g) = \emptyset$, (17) is trivial with $X' = X$, and by the equivalence (16) \Leftrightarrow (17), which has already been proved, we get that $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y}$ is a covering of X . It remains to show that the same covering property holds when $g \equiv +\infty$. For all $x \in X$ and $y \in Y$, we have $B^\circ g(y) = -\infty = b^\circ(y, x, g(x))$. Taking $y \in \mathcal{S}_x$, we get $y \in \partial^\circ g(x)$ by (11b), which shows that $\cup_{y \in Y} (\partial^\circ g)^{-1}(y) = X$ in this case, too. \square

We next mention some direct consequences of the continuity and coercivity assumptions.

LEMMA 3.21. *The kernel b is continuous in the second variable if, and only if, for all $x \in X$ and $\alpha \in \mathbb{R}$, $b(x, \cdot, \alpha)$ is upper semicontinuous (u.s.c.). In that case, $b(x, \cdot, \alpha)$ is continuous, for all $x \in X$ and $\alpha \in \mathbb{R} \cup \{+\infty\}$.*

PROOF. Proposition 2.3 shows that $b(x, \cdot, \alpha)$ is l.s.c. for all $x \in X$ and $\alpha \in \overline{\mathbb{R}}$. Hence, for all $x \in X$ and $\alpha \in \mathbb{R}$, $b(x, \cdot, \alpha)$ is u.s.c. if, and only if, it is continuous. Moreover, since, by (6), $b(x, \cdot, +\infty) \equiv -\infty$, $b(x, \cdot, +\infty)$ is always (trivially) continuous. \square

Note that the continuity assumption does *not* require that $b(x, \cdot, -\infty)$ is continuous or u.s.c. (indeed, in the special case when $b(x, \cdot, \alpha) = \bar{b}(x, \cdot) - \alpha$, we have $b(x, \cdot, -\infty) = \bar{b}(x, \cdot) + \infty$, which need not be u.s.c. if $\bar{b}(x, \cdot)$ is continuous and takes the value $-\infty$). The next lemma shows that assuming b or b° to be u.s.c. (or equivalently continuous) is the same:

LEMMA 3.22. *Let $x \in X$. Then $b(x, \cdot, \beta)$ is u.s.c. for all $\beta \in \mathbb{R}$ if, and only if, $b^\circ(\cdot, x, \alpha)$ is u.s.c. for all $\alpha \in \mathbb{R}$.*

PROOF. We already observed in the proof of Lemma 3.21 that $b(x, \cdot, +\infty)$ and $b^\circ(\cdot, x, +\infty)$ are u.s.c., so it is enough to show that

$$(26) \quad (b(x, \cdot, \beta) \text{ is u.s.c. } \forall \beta \in \mathbb{R} \cup \{+\infty\}) \iff (b^\circ(\cdot, x, \alpha) \text{ is u.s.c. } \forall \alpha \in \mathbb{R} \cup \{+\infty\}).$$

The left hand side of (26) is equivalent to

$$(27) \quad \{y \in Y \mid b(x, y, \beta) \geq \alpha\} \text{ is closed } \forall \alpha, \beta \in \mathbb{R} \cup \{+\infty\}.$$

Applying (20a), we get that (27) is equivalent to

$$\{y \in Y \mid b^\circ(y, x, \alpha) \geq \beta\} \text{ is closed } \forall \alpha, \beta \in \mathbb{R} \cup \{+\infty\},$$

which is exactly the upper semicontinuity of all the maps $b^\circ(\cdot, x, \alpha)$, with $\alpha \in \mathbb{R} \cup \{+\infty\}$. \square

LEMMA 3.23. *If b is continuous in the second variable, then for all maps $f \in \mathcal{F}$ such that $f(y) > -\infty$ for all $y \in Y$, the map $y \mapsto b(x, y, f(y))$ is u.s.c. for all $x \in X$.*

PROOF. We have to show that $\{y \in Y \mid b(x, y, f(y)) \geq \beta\}$ is closed, for all $\beta \in \mathbb{R} \cup \{+\infty\}$. Since $f(y) > -\infty$, for all $y \in Y$, (20a) yields $\{y \in Y \mid b(x, y, f(y)) \geq \beta\} = \{y \in Y \mid b^\circ(y, x, \beta) \geq f(y)\}$, which is closed since f is l.s.c. and $b^\circ(\cdot, x, \beta)$ is u.s.c. (by Lemma 3.21). \square

The following observation shows that we could have replaced “relatively compact” by “compact” in the definition of coercivity.

PROPOSITION 3.24. *If b is continuous in the second variable and coercive, then, for all $\alpha, \beta \in \mathbb{R}$, for all $x \in X$ and neighbourhoods V of x , $\{y \in Y \mid b_{x,V}^\alpha(y) \leq \beta\}$ is compact.*

PROOF. Since $b_{x,V}^\alpha$ is given by the sup in (13), we have $\{y \in Y \mid b_{x,V}^\alpha(y) \leq \beta\} = \cap_{z \in V} Y_z$, where $Y_z = \{y \in Y \mid b(z, y, b^\circ(y, x, \alpha)) \leq \beta\}$. By (2b), $Y_z = \{y \in Y \mid b^\circ(y, z, \beta) \leq b^\circ(y, x, \alpha)\}$, which is closed because $b^\circ(\cdot, z, \beta)$ is l.s.c. (by Theorem 2.1), and $b^\circ(\cdot, x, \alpha)$ is u.s.c. for $\alpha \in \mathbb{R}$ (by Lemma 3.22 and the continuity of b in the second variable). Therefore, $\cap_{z \in V} Y_z$, which is closed and relatively compact, is compact. \square

The proof of (14) \Rightarrow (17) in Theorem 3.5 relies on the following result:

THEOREM 3.25. *Let $f \in \mathcal{F}$. Assume that either Y is discrete, or b is continuous in the second variable and $f(y) > -\infty$ for all $y \in Y$. Then, if $f \in \mathcal{F}_c$, $\{\partial f(y)\}_{y \in \text{ldom}(f)}$ is a covering of $\text{udom}(Bf)$, and if b is coercive, $\{\partial f(y)\}_{y \in \text{ldom}(f)}$ is a covering of $\text{idom}(Bf)$.*

PROOF. We set $g = Bf$. We prove at the same time the two assertions of the theorem by setting $X_0 = \text{udom}(g)$ when $f \in \mathcal{F}_c$, and $X_0 = \text{idom}(g)$ when b is coercive. We thus need to prove that $X_0 \subset \cup_{y \in \text{ldom}(f)} \partial f(y)$. Since, by (23a), $\cup_{y \in Y \setminus \text{ldom}(f)} \partial f(y) = X \setminus \text{udom}(g)$, and since $X_0 \subset \text{udom}(g)$, it is sufficient to prove that $X_0 \subset \cup_{y \in Y} \partial f(y)$. We will prove:

$$(28) \quad x \in X_0 \implies \exists y \in Y, b(x, y, f(y)) = \sup_{y' \in Y} b(x, y', f(y')) .$$

Indeed, if (28) holds, then for all $x \in X_0$, there exists $y \in Y$ such that $b(x, y, f(y)) = Bf(x) = g(x)$ and since $X_0 \subset \text{udom}(g)$, $g(x) \neq -\infty$. Hence $b(x, y, f(y)) \neq -\infty$, whence $(x, y) \in \mathcal{S}$, which implies with $b(x, y, f(y)) = Bf(x)$ that $x \in \partial f(y)$. This shows that $X_0 \subset \cup_{y \in Y} \partial f(y)$.

To prove (28), it suffices to show that

$$(29) \quad \forall x \in X_0, \forall \alpha \in \mathbb{R}, L_\alpha(x) = \{y \in Y \mid b(x, y, f(y)) \geq \alpha\} \text{ is compact.}$$

Let us first prove that the sets $L_\alpha(x)$ are closed for all $x \in X$ and $\alpha \in \mathbb{R}$. When Y is discrete, this is trivial. Otherwise, by the assumptions of the theorem, $f(y) > -\infty$, for all $y \in Y$, and b is continuous in the second variable, therefore, by Lemma 3.23, $y \mapsto b(x, y, f(y))$ is an u.s.c. map for all $x \in X$. This implies again that the sets $L_\alpha(x)$ are closed for all $x \in X$ and $\alpha \in \mathbb{R}$.

It remains to show that the sets $L_\alpha(x)$ are relatively compact for all $x \in X_0$ and $\alpha \in \mathbb{R}$. By definition of \mathcal{F}_c , this holds trivially for any $X_0 \subset X$, when $f \in \mathcal{F}_c$. Let us finally assume that b is coercive and $X_0 = \text{idom}(g)$. Let $x \in \text{idom}(g)$ and $\alpha \in \mathbb{R}$. There exists $\beta \in \mathbb{R}$ such that $\limsup_{x' \rightarrow x} g(x') < \beta$, so there exists a neighbourhood V of x in X such that $\sup_{x' \in V} g(x') \leq \beta$. Then, by (20b):

$$b(x, y, f(y)) \geq \alpha \implies b^\circ(y, x, \alpha) \geq f(y) ,$$

and since

$$f(y) \geq B^\circ g(y) \geq b^\circ(y, z, g(z)) \geq b^\circ(y, z, \beta) \quad \forall z \in V ,$$

we obtain:

$$\begin{aligned} b(x, y, f(y)) \geq \alpha &\implies \forall z \in V, b^\circ(y, x, \alpha) \geq b^\circ(y, z, \beta) \\ &\implies \forall z \in V, \beta \geq b(z, y, b^\circ(y, x, \alpha)) , \end{aligned}$$

which shows that $L_\alpha(x) \subset \{y \in Y \mid b_{x,V}^\alpha(y) \leq \beta\}$. By the coercivity of b , the latter set is relatively compact, and thus $L_\alpha(x)$ is also relatively compact. This concludes the proof of (29). \square

PROOF OF (14) \Rightarrow (17) IN THEOREM 3.5. If (\mathcal{P}') has a solution $f \in \mathcal{F}$, then, by Lemma 3.18 and (2a), $f = B^\circ g$ is also a solution of (\mathcal{P}') . Fix $f = B^\circ g$. By Lemma 3.19, $(\cup_{y \in \text{idom}(f)} \partial f(y)) \cap X' \subset \cup_{y \in \text{idom}(B^\circ g)} (\partial^\circ g)^{-1}(y)$.

Let us first consider the case where Assumption (A6) holds. Then, applying Theorem 3.25 to f , we deduce that $\text{idom}(Bf) \cap X' \subset \cup_{y \in \text{idom}(B^\circ g)} (\partial^\circ g)^{-1}(y)$. Since it is clear that $X' \cap \text{idom}(g) \subset X' \cap \text{idom}(Bf)$, we get (17).

Let us finally consider the case where Assumption (A6)' holds. Then, applying Theorem 3.25 to f , we obtain that $\text{idom}(Bf) \cap X' \subset (\cup_{y \in \text{idom}(f)} \partial f(y)) \cap X' \subset \cup_{y \in \text{idom}(B^\circ g)} (\partial^\circ g)^{-1}(y)$. Moreover, it is easy to show that $X' \subset \text{idom}(g) \cup g^{-1}(-\infty)$ implies $X' \cap \text{idom}(g) \subset X' \cap \text{idom}(Bf)$, hence (17) follows. \square

PROOF OF COROLLARY 3.10. When Assumptions (A5)', and (A6), hold, the equivalence (14) \Leftrightarrow (16) in Theorem 3.5, for $X' = X$, yields the first assertion of the corollary. Assume now that $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y}$ is a covering of X . Then, for all $x \in X$, there exists $y \in \partial^\circ g(x)$. By (11b), $(x, y) \in \mathcal{S}$ and $B^\circ g(y) = b^\circ(y, x, g(x))$. Since $B^\circ g(y) > -\infty$ for all $y \in Y$, and $b^\circ(y, x, \cdot)$ is a decreasing bijection $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$, we deduce that $g(x) < +\infty$, for all $x \in X$. \square

4. Uniqueness of Solutions of $Bf = g$

4.1. Statement of the Uniqueness Results. To give a uniqueness result, we need some additional definitions.

DEFINITION 4.1. Let F be a map from a set Z to the set $\mathcal{P}(W)$ of all subsets of some set W , and let $Z' \subset Z$ and $W' \subset W$ be such that $\{F(z)\}_{z \in Z'}$ is a covering of W' . An element $y \in Z'$ is said *algebraically essential* with respect to the covering $\{F(z)\}_{z \in Z'}$ of W' if there exists $w \in W'$ such that $w \notin \cup_{z \in Z' \setminus \{y\}} F(z)$. When Z is a topological space, an element $y \in Z'$ is said *topologically essential* with respect to the covering $\{F(z)\}_{z \in Z'}$ of W' if for all open neighbourhoods U of y in Z' , there exists $w \in W'$ such that $w \notin \cup_{z \in Z' \setminus U} F(z)$. The covering of W' by $\{F(z)\}_{z \in Z'}$ is *algebraically (resp. topologically) minimal* if all elements of Z' are algebraically (resp. topologically) essential.

Algebraic minimality implies topological minimality. Both notions coincide if Z is a discrete topological space.

DEFINITION 4.2. Let $f \in \mathcal{F}$ and $X' \subset X$. We say that $y \in Y$ is an *exposed point* of f relative to X' if there exists $x \in X'$ such that $(x, y) \in \mathcal{S}$ and

$$b(x, y', f(y')) < b(x, y, f(y)) \quad \forall y' \in Y \setminus \{y\} .$$

When B is the Legendre-Fenchel transform and $X' = X$, this notion coincides with the definition given in [DZ93, Definition 2.3.3] of an exposed point of f . It is equivalent to the property that $(y, f(y))$ is an exposed point of the epigraph of f [Roc70, Sections 18 and 25]. We readily get from Definitions 3.3 and 4.2:

LEMMA 4.3. *Let $f \in \mathcal{F}$ and let $X' \subset \cup_{y \in Y} \partial f(y)$. An element $z \in Y$ is an exposed point of f relative to X' if, and only if, z is algebraically essential with respect to the covering $\{\partial f(y)\}_{y \in Y}$ of X' .*

DEFINITION 4.4. Let Z and W be topological spaces. We say that a map $h : Z \rightarrow W$ is *quasi-continuous* if for all open sets G of W , the set $h^{-1}(G)$ is *semi-open*, which means that $h^{-1}(G)$ is included in the closure of its interior.

See for instance [Neu89] for definitions and properties of quasi-continuous functions or multi-applications. If $h : X \rightarrow \overline{\mathbb{R}}$ is l.s.c., then h is quasi-continuous if, and only if, $h = \text{lsc}(\text{usc}(h))$, where lsc (resp. usc) means the l.s.c. (resp. u.s.c.) hull. The notion of quasi-continuous function, and the properties of l.s.c. or u.s.c. quasi-continuous functions have also been studied in [Sam02].

DEFINITION 4.5. We say that B is *regular* if for all $f \in \mathcal{F}$, Bf is l.s.c. on X and quasi-continuous on its domain, which means that the restriction of Bf to its domain is quasi-continuous for the induced topology.

The notion of regularity for B° is defined in the symmetric way. When X (resp. Y) is endowed with the discrete topology, B (resp. B°) is always regular. When $\mathcal{S} = X \times Y$ and $\{b(\cdot, y, \alpha)\}_{y \in Y, \alpha \in \mathbb{R}}$ is an equicontinuous family of functions, then Bf is continuous on X for any $f \in \mathcal{F}$, so B is regular. The Legendre-Fenchel transform on \mathbb{R}^n is regular (see Lemma 6.1 below).

We now state several uniqueness results, that we shall prove in Section 4.2.

THEOREM 4.6. *Let $X' \subset X$, and $g \in \mathcal{G}$. Assume that $\{(\partial^\circ g)^{-1}(y)\}_{y \in \text{ldom}(B^\circ g)}$ is a covering of $X' \cap \text{udom}(g)$, and denote by Z_a (resp. Z_t) the set of algebraically (resp. topologically) essential elements with respect to this covering. Make Assumptions (A5) or (A5)', and (A6) or (A6)'. Then Problem (\mathcal{P}') has a solution, and any solution f of (\mathcal{P}') satisfies*

$$(30) \quad f \geq B^\circ g, \quad \text{and} \quad f(y) = B^\circ g(y) \quad \text{for all } y \in Z_a.$$

If, in addition, $B^\circ g$ is quasi-continuous on its domain, and $\text{int}(Z_t)$ denotes the interior of Z_t , relatively to $\text{ldom}(B^\circ g)$, then any solution f of (\mathcal{P}') satisfies

$$(31) \quad f(y) = B^\circ g(y) \quad \text{for all } y \in \text{int}(Z_t).$$

THEOREM 4.7. *Let $X' \subset X$, and $g \in \mathcal{G}$. Consider the following statements:*

$$(32) \quad \text{Problem } (\mathcal{P}') \text{ has a unique solution,}$$

$$(33) \quad \{(\partial^\circ g)^{-1}(y)\}_{y \in \text{ldom}(B^\circ g)} \text{ is a topologically minimal covering of } X' \cap \text{udom}(g).$$

We have $(32) \Rightarrow (33)$. The implication $(32) \Rightarrow (33)$ holds if Assumptions (A5) or (A5)', and (A6) or (A6)', are satisfied. The equivalence $(32) \Leftrightarrow (33)$ holds if we assume in addition that $B^\circ g$ is quasi-continuous on its domain. In particular, this equivalence holds when Y is finite.

The topological minimality in (33) is a relaxation of algebraic minimality, which is a generalised differentiability condition. Indeed, if $\{(\partial^\circ g)^{-1}(y)\}_{y \in \text{ldom}(B^\circ g)}$ is a covering of $X' \cap \text{udom}(g)$, this covering is algebraically minimal if, and only if, for all $y \in \text{ldom}(B^\circ g)$, there is an $x \in X' \cap \text{udom}(g)$ such that $\partial^\circ g(x) = \{y\}$. This is in particular fulfilled when $\{(\partial^\circ g)(x)\}_{x \in X' \cap \text{udom}(g)}$ is a covering of $\text{ldom}(B^\circ g)$,

and for all $x \in X' \cap \text{udom}(g)$, $\partial^{\circ}g(x)$ is a singleton, a condition which, in the case where B is the Legendre-Fenchel transform, means that g is differentiable at x .

COROLLARY 4.8. *Consider $g \in \mathcal{G}$. Assume that Y is finite. Then the equation $Bf = g$ has a unique solution $f \in \mathcal{F}$, if, and only if, $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in \text{ldom}(B^{\circ}g)}$ is an algebraically minimal covering of $\text{udom}(g)$.*

COROLLARY 4.9. *Let $g \in \mathcal{G}$. Make Assumptions (A5)', and (A6). Assume in addition that $B^{\circ}g$ is quasi-continuous on its domain. Then the equation $Bf = g$ has a unique solution $f \in \mathcal{F}$, if, and only if, $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in \text{ldom}(B^{\circ}g)}$ is a topologically minimal covering of $\text{udom}(g)$.*

Since (32) implies that Problem (\mathcal{P}) has at most one solution, Theorem 4.7 yields a sufficient condition for the uniqueness of the solution of Problem (\mathcal{P}) . However, for Problem (\mathcal{P}) , the necessary uniqueness condition implied by Theorem 4.7 only holds when $B^{\circ}g \in \mathcal{F}_c$, or when $X = \text{idom}(g) \cup g^{-1}(-\infty)$, or when $X = \text{dom}(\partial^{\circ}g)$. To give a more specific uniqueness result for Problem (\mathcal{P}) , we shall use the following condition:

there exists a basis \mathcal{B} of neighborhoods such that

$$(\mathcal{C}) : \quad \forall U \in \mathcal{B}, \exists \varepsilon > 0, \forall x \in X, \sup_{y \in U \cap \mathcal{S}_x, \alpha \in \mathbb{R}} (b(x, y, \alpha) - b(x, y, \alpha + \varepsilon)) < +\infty.$$

Condition (\mathcal{C}) is satisfied in particular when $\{b(x, y, \cdot)\}_{x \in X, y \in Y}$ is a family of β -Hölder continuous functions (for $0 < \beta \leq 1$), uniformly in $y \in U$, for all small enough open sets U , or if $\{b(x, y, \cdot)\}_{x \in X, y \in U}$ is an equicontinuous family, for all small enough open sets U . In particular, condition (\mathcal{C}) is satisfied when $b(x, y, \alpha) = b(x, y) - \alpha$ or when $b(x, y, \alpha) = -(|\langle x, y \rangle| + 1)\alpha$.

THEOREM 4.10. *Let $g \in \mathcal{G}$. Then the existence and uniqueness of a solution of Problem (\mathcal{P}) implies (33), if one of the following assertions is satisfied:*

- (1) $X' = \text{dom}(\partial^{\circ}g)$, and Assumption (A5) holds;
- (2) $X' = \text{dom}(\partial^{\circ}g)$, Y is locally compact, and Assumption (A5)' holds;
- (3) $X' = \text{idom}(g)$, b is coercive, B is regular, Assumption (A5)' holds, $\text{dom}(g)$ is included in the closure of $\text{idom}(g)$, and either Y is locally compact or condition (\mathcal{C}) holds.

4.2. Proofs of the Uniqueness Results. Let us first state a general property of quasi-continuous functions.

LEMMA 4.11. *Let $f \in \text{lsc}(Y, \overline{\mathbb{R}})$, and let $h : Y \rightarrow \overline{\mathbb{R}}$ be a quasi-continuous map. Then the set $V = \{y \in Y \mid h(y) < f(y)\}$ is semi-open. In particular, if V is non-empty, V has a non-empty interior.*

PROOF. It suffices to consider the case when V is non-empty. Let $z \in V$. There exists $a \in \mathbb{R}$ such that $h(z) < a < f(z)$. Consider $V_1 = \{y \in Y \mid h(y) < a\}$, U_1 the interior of V_1 , and $U_2 = \{y \in Y \mid a < f(y)\}$. We have $z \in V_1 \cap U_2 \subset V$. Since h is quasi-continuous, V_1 is semi-open, hence V_1 is included in the closure of U_1 , that we denote by $\overline{U_1}$. Since f is l.s.c., U_2 is open. We have $z \in V_1 \cap U_2 \subset \overline{U_1} \cap U_2 \subset \overline{U_1 \cap U_2}$, and since $U_1 \cap U_2$ is open and included in V , we get that z belongs to the closure of the interior of V . \square

PROOF OF THEOREM 4.6. Note first that since, when $Bf \leq g$, the equation $Bf = g$ on X' is equivalent to the equation $Bf = g$ on $X' \cap \text{udom}(g)$, we can assume

without restriction of generality that $X' \subset \text{udom}(g)$ in the proofs of Theorems 4.6 and 4.7.

Let $Y' = \text{ldom}(B^\circ g)$, assume that $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y'}$ is a covering of X' , and denote by Z_a (resp. Z_t) the set of algebraically (resp. topologically) essential elements with respect to this covering. Since (17) holds, it follows from Theorem 3.5, that (\mathcal{P}') has a solution $f \in \mathcal{F}$, and by Lemma 3.18 and (2a), $B^\circ g$ is necessarily another solution and $B^\circ g \leq f$.

Let $f \in \mathcal{F}$ be a solution of (\mathcal{P}') and denote by $F = \{y \in Y' \mid B^\circ g(y) = f(y)\}$ and by V its complement in Y' . Since, $B^\circ g \leq f$, $V = \{y \in Y' \mid B^\circ g(y) < f(y)\}$. We claim that

$$(34) \quad \forall x \in X' \exists y \in F \text{ such that } x \in (\partial^\circ g)^{-1}(y).$$

If (34) is proved, then the following holds

$$(35) \quad Z_a \subset F \text{ and } Z_t \subset \overline{F},$$

where \overline{F} denotes the closure of F , relatively to Y' . Indeed, let us first consider $z \in Z_a$. Since z is algebraically essential with respect to the covering $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y'}$ of X' , there exists $x \in X'$ such that

$$(36) \quad x \in (\partial^\circ g)^{-1}(z) \setminus \bigcup_{y \in Y' \setminus \{z\}} (\partial^\circ g)^{-1}(y).$$

Moreover, by (34), there exists $y \in F$ such that $x \in (\partial^\circ g)^{-1}(y)$. Using (36), this implies that $y = z$ and $z \in F$, which shows $Z_a \subset F$. Let us now consider $z \in Z_t$. Since z is topologically essential with respect to the covering $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y'}$ of X' , for all open neighbourhoods U of z in Y' , there exists $x_U \in X'$ such that

$$(37) \quad x_U \in \bigcup_{y \in U} (\partial^\circ g)^{-1}(y) \setminus \bigcup_{y' \in Y' \setminus U} (\partial^\circ g)^{-1}(y').$$

Moreover, by (34), there exists $y_U \in F$ such that $x_U \in (\partial^\circ g)^{-1}(y_U)$. Using (37), this implies that $y_U \in U$. We have thus proved that for all open neighbourhoods U of z in Y' , there exists $y_U \in U \cap F$, which means that $z \in \overline{F}$, and shows $Z_t \subset \overline{F}$.

Now from (35), we get (30). If Assumption (A5) holds, that is, if Y is discrete, $Z_a = Z_t$ and thus (31) holds trivially by (30). Otherwise, we deduce from Assumption (A5)' that $Y' = \text{ldom}(B^\circ g) = \text{dom}(B^\circ g)$. Moreover, if $B^\circ g$ is quasi-continuous on its domain, then, since f is l.s.c. on Y , and a fortiori on Y' , we obtain, by Lemma 4.11, that V is semi-open in Y' . It follows that its complement in Y' , F , contains the interior of its closure \overline{F} , relatively to Y' . Using (35), this yields $\text{int}(Z_t) \subset F$, which means precisely that (31) holds.

Let us prove (34). When f is a solution of (\mathcal{P}') , $X' \subset \text{udom}(g)$ implies $X' \subset \text{udom}(Bf)$, $X' \subset \text{idom}(g)$ implies $X' \subset \text{idom}(Bf)$, and $B^\circ g \in \mathcal{F}_c$ implies $f \in \mathcal{F}_c$ (since $f \geq B^\circ g$). Hence, Theorem 3.25 shows that $X' \subset \bigcup_{y \in \text{ldom}(f)} \partial f(y)$. Since $B^\circ g \leq f$ implies that $\text{ldom}(f) \subset \text{ldom}(B^\circ g) = Y'$, we get that $X' \subset \bigcup_{y \in Y'} \partial f(y)$. Hence, for all $x \in X'$, there exists $y \in Y'$ such that $x \in \partial f(y)$. Since then $\partial f(y) \cap X' \neq \emptyset$, Lemma 3.18 shows that $f(y) = B^\circ g(y)$ and $\partial f(y) \cap X' \subset (\partial^\circ g)^{-1}(y)$. Hence, $y \in F$ and $x \in (\partial^\circ g)^{-1}(y)$, which shows (34). \square

We now prove the different assertions of Theorem 4.7.

PROOF OF (32,17) \Rightarrow (33) IN THEOREM 4.7. We assume, as in the above proof, that $X' \subset \text{udom}(g)$. Set $Y' = \text{ldom}(B^\circ g)$. Assume that (32) and (17) hold, which means that (\mathcal{P}') has a unique solution and $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y'}$ is a covering of X' . Assume by contradiction that this covering is not topologically minimal, i.e., that

there exists an open set U of Y such that $U \cap Y' \neq \emptyset$, and such that for all $x \in X'$, there exists $y \in Y' \setminus U$ such that $x \in (\partial^{\circ}g)^{-1}(y)$, which means, by (11b), that $(x, y) \in \mathcal{S}$ and $B^{\circ}g(y) = b^{\circ}(y, x, g(x))$. Then $g(x) = b(x, y, B^{\circ}g(y))$ and, since $g \geq BB^{\circ}g$, we get:

$$(38) \quad g(x) = \sup_{y \in Y \setminus U} b(x, y, B^{\circ}g(y)) \quad \forall x \in X'.$$

To contradict the uniqueness of the solution of (\mathcal{P}') , it suffices to construct a map $f \in \mathcal{F}$ such that $f \neq B^{\circ}g$ and

$$(39) \quad f \geq B^{\circ}g, \quad f = B^{\circ}g \text{ on } Y \setminus U.$$

Indeed, for any function f satisfying (39), we have $Bf \leq g$, and, by (38), $Bf \geq g$ on X' , hence f is a solution of (\mathcal{P}') . The function $f = B^{\circ}g$ satisfies trivially $f \in \mathcal{F}$ and (39). Defining f by $f = B^{\circ}g$ on $Y \setminus U$ and $f = +\infty$ on U , we obtain that $f \in \mathcal{F}$, f satisfies (39) and since $Y' \cap U \neq \emptyset$, $f \neq B^{\circ}g$, which concludes the proof. \square

PROOF OF $(32) \Rightarrow (33)$ IN THEOREM 4.7. If (32) holds, we get in particular that Problem (\mathcal{P}') has a solution, and we deduce (17) from Theorem 3.5, using the Assumptions of Theorem 4.7. From the implication $(32, 17) \Rightarrow (33)$ that we already proved, we obtain $(32) \Rightarrow (33)$. \square

PROOF OF $(33) \Rightarrow (32)$ IN THEOREM 4.7. The assumptions for this implication, in Theorem 4.7, imply that the assumptions of Theorem 4.6 are satisfied. They also imply that any element of $\text{ldom}(B^{\circ}g)$ is topological essential for the covering $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in \text{ldom}(B^{\circ}g)}$ of $X' \cap \text{udom}(g)$. This means that $Z_t = \text{ldom}(B^{\circ}g)$ in Theorem 4.6, thus $\text{int}(Z_t) = Z_t = \text{ldom}(B^{\circ}g)$, hence the conclusions of Theorem 4.6 imply that (\mathcal{P}') has a solution and that any solution f of (\mathcal{P}') satisfies $f \geq B^{\circ}g$ and $f(y) = B^{\circ}g(y)$ for all $y \in \text{ldom}(B^{\circ}g)$. Since, $f \geq B^{\circ}g$, then $f(y) = B^{\circ}g(y) = +\infty$ for all $y \in Y \setminus \text{ldom}(B^{\circ}g)$. Therefore, $f = B^{\circ}g$. \square

We now prove Theorem 4.10.

PROOF OF THEOREM 4.10 IN CASES (1) AND (2). Assume that $Bf = g$ has a unique solution $f \in \mathcal{F}$ and that Condition (1) or Condition (2) holds. In particular, $BB^{\circ}g = g$ and $X' = \text{dom}(\partial^{\circ}g)$. Set $Y' = \text{ldom}(B^{\circ}g)$. By the equivalence $(15) \Leftrightarrow (17)$ in Theorem 3.5, $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in Y'}$ is a covering of $X' \cap \text{udom}(g)$. We shall show that this covering is topologically minimal. Arguing by contradiction, and using the same arguments as in the proof of the implication $(32, 17) \Rightarrow (33)$ in Theorem 4.7, we obtain that there exists an open set U of Y such that $U \cap Y' \neq \emptyset$ and (38) holds. Since Y is locally compact (this holds trivially in Case (1), and by assumption in Case (2)), possibly after replacing U by an open subset, we can assume that U is relatively compact, which means that its closure is compact. Taking $f = B^{\circ}g$ on $Y \setminus U$ and $f = +\infty$ on U , we deduce, as in the proof of $(32, 17) \Rightarrow (33)$, that $f \neq B^{\circ}g$, $Bf \leq g$ and $Bf = g$ on X' (we obtain $Bf \leq g$ and $Bf = g$ on $X' \cap \text{udom}(g)$, and since $Bf \leq g \Rightarrow Bf = g$ on $g^{-1}(-\infty)$, we get $Bf = g$ on X'), hence

$$(40) \quad S \stackrel{\text{def}}{=} \{x \in X \mid Bf(x) \neq g(x)\} = \{x \in X \mid Bf(x) < g(x)\} \subset X \setminus X'.$$

It remains to check that $S = \emptyset$, in order to contradict the uniqueness of the solution f of $Bf = g$. Assume by contradiction that $S \neq \emptyset$. If $x \in S$, it follows

from (39), that $\sup_{y \in Y \setminus U} b(x, y, B^\circ g(y)) \leq Bf(x) < g(x)$. Since $BB^\circ g = g$, we get

$$g(x) = \sup_{y \in U} b(x, y, B^\circ g(y)) .$$

Since U has a compact closure \overline{U} , and the function $y \mapsto b(x, y, B^\circ g(y))$ is u.s.c. (this holds trivially in Case (1), and this follows from Lemma 3.23 and Assumption (A5)', in Case (2)),

$$(41) \quad \exists y \in \overline{U} \text{ such that } g(x) = b(x, y, B^\circ g(y)) .$$

Since $g(x) > -\infty$, we get that $(x, y) \in \mathcal{S}$ and, by (11a) and Proposition 3.16, $x \in \partial B^\circ g(y) = (\partial^\circ g)^{-1}(y)$, which implies that $x \in \text{dom}(\partial^\circ g) = X'$. By (40), we get a contradiction. \square

PROOF OF THEOREM 4.10 IN CASE (3). Assume that $Bf = g$ has a unique solution $f \in \mathcal{F}$ and that Condition (3) holds. In particular, $BB^\circ g = g$. Set $Y' = \text{ldom}(B^\circ g) = \text{dom}(B^\circ g)$ and $X' = \text{idom}(g)$. Theorem 3.5 shows that $\{(\partial^\circ g)^{-1}(y)\}_{y \in Y'}$ is a covering of $X' \cap \text{udom}(g) = \text{idom}(g)$. We shall show that this covering is minimal.

Arguing by contradiction, and using the same arguments as in the proof of the implication (32,17) \Rightarrow (33) in Theorem 4.7, we obtain that there exists an open set U of Y such that $U \cap Y' \neq \emptyset$ and (38) holds. For any given basis of open neighbourhoods in Y , \mathcal{B} , possibly after replacing U by an open subset, we can assume that $U \in \mathcal{B}$ and $U \cap Y' \neq \emptyset$. We shall take either \mathcal{B} as in condition (C), or \mathcal{B} as the basis of relatively compact open sets.

Fix $\varepsilon > 0$ and consider the l.s.c. finite function $w : Y \rightarrow [0, 1]$ given by $w(y) = 0$ for $y \in Y \setminus U$ and $w(y) = \varepsilon$ for $y \in U$. Taking $f = B^\circ g + w$, we get that f is l.s.c., f satisfies (39), and $f \neq B^\circ g$ since $f(y) = B^\circ g(y) + \varepsilon > B^\circ g(y)$ for $y \in \text{dom}(B^\circ g) \cap U = Y' \cap U \neq \emptyset$. As in the proof of the implication (32,17) \Rightarrow (33), we deduce that $Bf \leq g$, $Bf = g$ on X' , hence (40) holds and it remains to show that $S = \emptyset$.

Assume by contradiction that $S \neq \emptyset$. We first prove that

$$(42a) \quad \text{ldom}(g) \subset \text{ldom}(Bf) ,$$

$$(42b) \quad \text{udom}(g) \subset \text{udom}(Bf) .$$

Indeed, (42a) follows from $Bf \leq g$. Let $x \in \text{udom}(g)$, hence $g(x) > -\infty$. Since $g(x) = \sup_{y \in Y} b(x, y, B^\circ g(y))$, there exists $y \in Y$ such that $b(x, y, B^\circ g(y)) > -\infty$, hence $(x, y) \in \mathcal{S}$ and $B^\circ g(y) < +\infty$, then $f(y) \leq B^\circ g(y) + \varepsilon < +\infty$ and $Bf(x) \geq b(x, y, f(y)) > -\infty$, which concludes the proof of (42b).

Since $S \subset \text{ldom}(Bf) \cap \text{udom}(g)$, we deduce from (42b), that $S \subset \text{dom}(Bf)$. Since B is regular, Bf and g are l.s.c. on X and quasi-continuous on their domain. Hence, by Lemma 4.11, S is semi-open relatively to $\text{dom}(Bf)$. Since $S \neq \emptyset$, S has a nonempty interior relatively to $\text{dom}(Bf)$. This means that there exists an open set V of X such that

$$(43) \quad \emptyset \neq V \cap \text{dom}(Bf) \subset S .$$

By (43) and (42), we get

$$(44) \quad V \cap \text{dom}(g) \subset S .$$

If we know that $V \cap \text{dom}(g) \neq \emptyset$, then since we assumed that $\text{idom}(g)$ is dense in $\text{dom}(g)$, we get $V \cap \text{idom}(g) \neq \emptyset$, so by (44), $S \cap \text{idom}(g) \neq \emptyset$, i.e. $S \cap X' \neq \emptyset$, which contradicts (40). It remains to show that $V \cap \text{dom}(g) \neq \emptyset$.

If Y is locally compact, the arguments of the proof of Theorem 4.10 in Case (2) show that (41) holds for all $x \in S$. Since $B^\circ g(y) > -\infty$ for all $y \in Y$, we deduce that $g(x) < +\infty$, hence $S \subset \text{ldom}(g)$. Since we also have $S \subset \text{udom}(g)$, we get $S \subset \text{dom}(g)$ and by (43) and (42), $V \cap \text{dom}(g) = V \cap \text{dom}(Bf) \neq \emptyset$.

Otherwise, b satisfies Condition (\mathcal{C}) , and if $U \in \mathcal{B}$ and $\varepsilon > 0$ is chosen as in (\mathcal{C}) , we get that for all $x \in X$,

$$\begin{aligned} g(x) \geq Bf(x) &= \sup_{y \in Y, B^\circ g(y) < +\infty} b(x, y, B^\circ g(y) + w(y)) \\ &\geq \sup_{y \in \mathcal{S}_x, B^\circ g(y) < +\infty} b(x, y, B^\circ g(y)) \\ &\quad + \inf_{y \in \mathcal{S}_x, B^\circ g(y) < +\infty} (b(x, y, B^\circ g(y) + w(y)) - b(x, y, B^\circ g(y))) \\ &\geq g(x) + \inf_{y \in \mathcal{S}_x \cap U, B^\circ g(y) \in \mathbb{R}} (b(x, y, B^\circ g(y) + \varepsilon) - b(x, y, B^\circ g(y))) \\ &\geq g(x) + \inf_{y \in \mathcal{S}_x \cap U, \alpha \in \mathbb{R}} (b(x, y, \alpha + \varepsilon) - b(x, y, \alpha)). \end{aligned}$$

By (\mathcal{C}) , we obtain $\text{dom}(g) = \text{dom}(Bf)$, which shows, by (43), that $V \cap \text{dom}(g) \neq \emptyset$. \square

5. Algorithmic Issues

When X and Y are finite, and when the kernels b and b° are given in an effective way, Corollaries 3.7 and 4.8 yield an algorithm, which extends Zimmermann's algorithm, to solve the equation $Bf = g$ and to decide the uniqueness of its solution. Let us illustrate this algorithm by taking $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. Consider the kernel b and the map g given by the following table

$$(45) \quad b : \begin{pmatrix} x_1 & y_1 & y_2 & y_3 \\ x_2 & -\lambda & 4 - 3\lambda & 2 - \lambda \\ & -\text{sgn}(\lambda)\lambda^2 & 3 - \lambda & -\lambda \end{pmatrix}, \quad g : \begin{pmatrix} x_1 & 8 \\ x_2 & 6 \end{pmatrix},$$

which means for instance that $b(x_1, y_2, \lambda) = 4 - 3\lambda$ and $g(x_1) = 8$. (We denote by $\text{sgn}(\lambda) \in \{0, \pm 1\}$ the sign of a scalar λ .) Let $B : \overline{\mathbb{R}}^Y \rightarrow \overline{\mathbb{R}}^X$ denote the (dual) functional Galois connection with kernel b . Assumptions (A1–A4) are clearly satisfied with $\mathcal{S} = X \times Y$. Then the kernel of B° is

$$b^\circ : \begin{pmatrix} x_1 & x_2 \\ y_1 & -\lambda & -\text{sgn}(\lambda)\sqrt{|\lambda|} \\ y_2 & (4 - \lambda)/3 & 3 - \lambda \\ y_3 & 2 - \lambda & -\lambda \end{pmatrix},$$

and $B^\circ g$ is given by:

$$(46) \quad B^\circ g : \begin{pmatrix} x_1 & x_2 \\ y_1 & -8 & \vee & -\sqrt{6} \\ y_2 & (4 - 8)/3 & \vee & 3 - 6 \\ y_3 & 2 - 8 & \vee & -6 \end{pmatrix} = \begin{pmatrix} -\sqrt{6} \\ -4/3 \\ -6 \end{pmatrix},$$

where we underlined the terms which determine the maximum (recall that \vee denotes the sup law). By (12b), the sets $(\partial^\circ g)^{-1}(y_j)$ can be read directly from (46) by choosing, for each row y_j , the x_i variables corresponding to the underlined terms:

$$(\partial^\circ g)^{-1}(y_1) = \{x_2\}, \quad (\partial^\circ g)^{-1}(y_2) = \{x_1\}, \quad (\partial^\circ g)^{-1}(y_3) = \{x_1, x_2\}.$$

Since the union of these subsets is equal to $X = \{x_1, x_2\}$, it follows from Corollary 3.7 that $f = B^\circ g$ is a solution of $Bf = g$. It follows from Corollary 4.8 that this solution is not unique, because the covering $\{(\partial^\circ g)^{-1}(y_j)\}_{1 \leq j \leq 3}$ of X is not minimal: for instance, $\{(\partial^\circ g)^{-1}(y_3)\}$ is a subcovering of X , which reflects the fact that setting $f(y_1) = f(y_2) = +\infty$ and $f(y_3) = -6$ yields another solution of $Bf = g$.

More generally, a minimal covering of a set of cardinality n must consist of at most n sets, which implies that when X and Y are finite, the number of elements of Y , i.e. the number of “scalar unknowns”, must not exceed the number of elements of X , i.e. the number of “scalar equations”, for the solution of $Bf = g$ to be unique.

To show a uniqueness case, consider the restriction $B_{1,2} : \overline{\mathbb{R}}^{\{y_1, y_2\}} \rightarrow \overline{\mathbb{R}}^X$, which is obtained by specialising B to those f such that $f(y_3) = +\infty$. Then the covering $\{(\partial^\circ g)^{-1}(y_j)\}_{1 \leq j \leq 2}$ of X is minimal, which shows that setting $f(y_1) = -\sqrt{6}$, $f(y_2) = -4/3$ yields the only solution of $B_{1,2}f = g$.

To illustrate the case where $Bf = g$ has no solution, consider:

$$g' : \begin{matrix} x_1 \\ x_2 \end{matrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \quad \text{with} \quad B^\circ g' : \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \begin{pmatrix} -3 & \vee & \frac{\sqrt{3}}{3} \\ (4-3)/3 & \vee & \frac{3+3}{3} \\ 2-3 & \vee & \underline{3} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 6 \\ 3 \end{pmatrix}.$$

We see from Corollary 3.7 that $Bf = g'$ has no solution, because $\bigcup_{1 \leq j \leq 3} (\partial^\circ g')^{-1}(y_j) = \{x_2\}$ is not a covering of X .

Finally, let us interpret these computations in geometric terms. For each $1 \leq j \leq 3$, denote by B_j the restriction of B , $\overline{\mathbb{R}}^{\{y_j\}} \rightarrow \overline{\mathbb{R}}^X$, which is obtained by specialising B to those f such that $f(y_k) = +\infty$ for $k \neq j$, so that the corresponding kernels b_j are given by:

$$b_1 : \begin{matrix} y_1 \\ x_1 \\ x_2 \end{matrix} \begin{pmatrix} -\lambda \\ 1 \\ -\text{sgn}(\lambda)\lambda^2 \end{pmatrix}, \quad b_2 : \begin{matrix} y_2 \\ x_1 \\ x_2 \end{matrix} \begin{pmatrix} 4-3\lambda \\ 1 \\ 3-\lambda \end{pmatrix}, \quad b_3 : \begin{matrix} y_3 \\ x_1 \\ x_2 \end{matrix} \begin{pmatrix} 2-\lambda \\ 1 \\ -\lambda \end{pmatrix}.$$

The (set of finite points of the) image of the operator B_1 is the curve $\lambda \rightarrow \text{sgn}(\lambda)\lambda^2$ which is depicted on Figure 2. The image of B_2 (resp. B_3) is the line with slope $1/3$ (resp. 1) on the figure. The image of B can be computed readily from the images of B_j : since $Bf = B_1f(y_1) \vee B_2f(y_2) \vee B_3f(y_3)$, the image of B is the sup-subsemilattice of $\overline{\mathbb{R}}^X$ generated by the images of B_1, B_2, B_3 , which corresponds to the gray region on Figure 2.

Now, for each dual functional Galois connection (B, B°) , observe that $BB^\circ g$ is the maximum element of the image of B which is below g . Thus, $P = BB^\circ$ is a nonlinear projector on the image of B , and for $j = 1, 2, 3$, consider the nonlinear projector $P_j = B_j B_j^\circ$ on the image of B_j . By definition of Galois connections, $P_j g(x) = b(x, y_j, B^\circ g(y_j))$, thus $P = \sup_{1 \leq j \leq 3} P_j$. The element g , and its image by the projectors P_j , are shown on Figure 2 (and can be computed directly from the figure). For each $1 \leq j \leq 3$, the set $(\partial^\circ g)^{-1}(y_j)$ represents the subset of elements

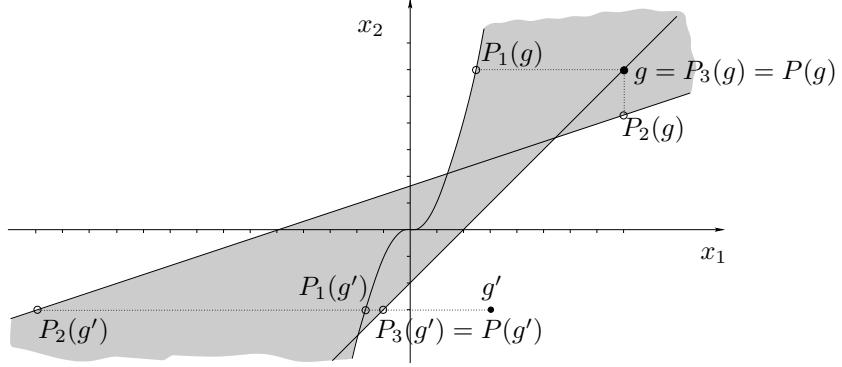


FIGURE 2. Image of the functional Galois connection (45).

x_i of $\{x_1, x_2\}$ such that $(P_j g)(x_i) = g(x_i)$. Thus, the covering condition $X \subset \bigcup_{1 \leq j \leq 3} (\partial^\circ g)^{-1}(y_j)$ is nothing but a combinatorial rephrasing of $g = \sup_{1 \leq j \leq 3} P_j g$.

6. Some Examples of Moreau Conjugacies

We give now some applications of the results of Sections 3 and 4 to the case of the Moreau conjugacies B and B° given by (7,9), for a kernel \bar{b} taking only finite values. Since $\mathcal{S} = X \times Y$, we have $B^\circ g(y) > -\infty$ for all $y \in Y$ and $g \in \mathcal{G}$ such that $g \not\equiv +\infty$.

6.1. The Legendre-Fenchel Transform. Let us consider the case where $X = Y = \mathbb{R}^n$ and $B = B^\circ$ is the Legendre-Fenchel transform, that is B and b are given by (7,9), with $\bar{b}(x, y) = \langle x, y \rangle$. We have already shown in Section 3.1 that b is continuous in the second variable and coercive. We also have:

LEMMA 6.1. *The Legendre-Fenchel transform on \mathbb{R}^n is regular.*

PROOF. We need to show that for any function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, $g = f^*$ is l.s.c. on \mathbb{R}^n and is quasi-continuous on its domain $\text{dom}(g)$. We know that g is either $\equiv +\infty$, or $\equiv -\infty$ or a l.s.c. proper convex function. Hence, it is l.s.c. and in the first two cases, the domain of g is empty. In the last case, since $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is l.s.c. on \mathbb{R}^n , and a fortiori on $\text{dom}(g)$, it is sufficient to prove that $g = \text{lsc}(\text{usc}(g))$ where lsc and usc envelopes are applied to the restrictions to $\text{dom}(g)$. Moreover, since g is l.s.c., we get $g \leq \text{lsc}(\text{usc}(g))$, hence it is sufficient to prove that $g \geq \text{lsc}(\text{usc}(g))$. The following properties of l.s.c. proper convex functions can be found in [Roc70]: g is continuous in the relative interior of $\text{dom}(g)$, that we denote by $\text{ridom}(g)$ (recall that the relative interior of a convex set is the interior of the set for the topology of the affine hull of the set), for any affine line L , the restriction of g to L is continuous on its domain $\text{dom}(g) \cap L$, and $\text{ridom}(g) \cap L$ is dense in $\text{dom}(g) \cap L$. From this, we get that $\text{usc}(g) = g$ on $\text{ridom}(g)$. Let us fix $x_0 \in \text{ridom}(g)$. For all $x \in \text{dom}(g)$, take the affine line L containing x_0 and x . Since there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{ridom}(g) \cap L$ converging to x and since g is continuous on $\text{dom}(g) \cap L$, we obtain that $g(x) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} \text{usc}(g)(x_n) \geq \text{lsc}(\text{usc}(g))(x)$, which finishes the proof. \square

Using Theorems 3.5 and 4.6, we get

PROPOSITION 6.2. *Let g be a l.s.c. proper convex function on \mathbb{R}^n . Then $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g^*)}$ is a covering of $\text{idom}(g)$. Let Z_a (resp. Z_t) be the set of algebraically (resp. topologically) essential elements with respect to this covering, and let $\text{int}(Z_t)$ denotes the interior of Z_t , relatively to $\text{dom}(g^*)$. Then*

$$\begin{aligned} & (f^* \leq g \text{ and } f^*(x) = g(x) \text{ for all } x \in \text{idom}(g)) \\ \implies & (g^* \leq f \text{ and } f(y) = g^*(y) \text{ for all } y \in Z_a \cup \text{int}(Z_t)). \end{aligned}$$

Since Y is locally compact, and for any l.s.c. proper convex function on \mathbb{R}^n such that $\text{dom}(g)$ has a nonempty interior, $\text{idom}(g)$, the set $\text{dom}(g)$ is included in the closure of $\text{idom}(g)$ [Roc70, Theorem 6.3], we can also apply Theorem 4.10. We deduce:

PROPOSITION 6.3. *Let g be a l.s.c. proper convex function on \mathbb{R}^n such that $\text{idom}(g) \neq \emptyset$. The following statements are equivalent:*

- (47) $(f^* \leq g \text{ and } f^*(x) = g(x) \text{ for all } x \in \text{idom}(g)) \implies f = g^*;$
- (48) $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g^*)}$ is a topologically minimal covering of $\text{idom}(g)$;
- (49) $f^* = g \implies f = g^*.$

The following classical notion is intermediate between algebraic and topological minimality: a l.s.c. proper convex function g on \mathbb{R}^n is *essentially smooth* if $\text{idom}(g) \neq \emptyset$, g is differentiable in $\text{idom}(g)$, and the norm of the differential of g at x tends to infinity, when x goes to the boundary of $\text{dom}(g)$, see [Roc70, Section 26]. A l.s.c. proper convex function f on \mathbb{R}^n is *essentially strictly convex* if the restriction of f to any affine line (or segment) in $\text{dom}(f)$ is strictly convex. A l.s.c. proper convex function g is essentially smooth if, and only if, its conjugate g^* is essentially strictly convex [Roc70, Theorem 26.3]. The following result, which can be compared with [Roc70, Corollary 26.4.1], is a corollary of Proposition 6.3. It is the underlying argument of the Gärtner-Ellis theorem and it is explicitly used in [OV95, Theorem 4.1 (c)], in [Gul03, Theorems 4.7 and 5.3], and in [Puh94, Lemmas 3.2 and 3.5].

COROLLARY 6.4. *Let g be an essentially smooth l.s.c. proper convex function on \mathbb{R}^n . If f is a l.s.c. function such that $f^* \leq g$ and $f^*(x) = g(x)$ for all $x \in \text{idom}(g)$, then $f = g^*$. In particular, g has a unique preimage by the Legendre-Fenchel transform.*

PROOF. First, since $f^* \leq g$ implies $f \geq g^*$, so $f = g^*$ outside $\text{dom}(g^*)$, one can replace Y by the affine hull of $\text{dom}(g^*)$, so that $\text{idom}(g^*)$ is the relative interior of $\text{dom}(g^*)$ (and is thus nonempty). The conditions on the differentials of g imply that $\partial g(x)$ is a singleton when $x \in \text{idom}(g)$ and is empty elsewhere (see [Roc70, Theorem 26.1]). Hence, applying Theorem 3.25 to g , we get that for all $y \in \text{idom}(g^*)$ there exists $x \in \text{idom}(g) = \text{dom}(g)$ such that $y \in \partial g(x)$. Since $\partial g(x) \neq \emptyset$, we get that $x \in \text{idom}(g)$, and $\{y\} = \partial g(x)$, hence any $y \in \text{idom}(g^*)$ cannot be removed in the covering of $\text{idom}(g)$ by $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g^*)}$. Moreover, since $\text{dom}(g^*)$ is included in the closure of $\text{idom}(g^*)$, any open set U of $\text{dom}(g^*)$ contains a point $y \in \text{idom}(g^*)$, so U cannot be removed in the covering of $\text{idom}(g)$ by $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g^*)}$. This shows that the covering of $\text{idom}(g)$ by $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(g^*)}$ is topologically minimal. The implication (48) \Rightarrow (47) in Proposition 6.3 yields the result of the corollary. \square

EXAMPLE 6.5. The following function g satisfies (48) and thus (47), but is not essentially smooth: consider $X = Y = \mathbb{R}^2$, $g = f^*$ where $f : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$, with $f(y) = y_1^2(y_2^2 + 3)$ if $|y_2| \leq 1$ and $f(y) = +\infty$ elsewhere. Indeed, since f is l.s.c. and convex, $f = g^*$. Since f is not strictly convex on $y_1 = 0$, g is not essentially smooth. If f is essentially strictly convex in a neighbourhood of $y \in \text{dom}(f)$, the point y cannot be removed in the covering of $\text{dom}(\partial g)$ by $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(f)}$. Then, since $\text{idom}(g) = \text{dom}(\partial g) = \text{dom}(g) = \mathbb{R}^2$ and the loss of strict convexity of f occurs only on a line, any open set of $\text{dom}(f)$ intersects the “part” of $\text{idom}(f)$ where f is essentially strictly convex. This implies that the covering of $\text{idom}(g) = \mathbb{R}^2$ by $\{(\partial g)^{-1}(y)\}_{y \in \text{dom}(f)}$ is topologically minimal.

6.2. Quadratic Kernels. Let us consider the case where $Y = X = \mathbb{R}^n$ and b is given by (9) with $\bar{b}(x, y) = b_a(x, y) := \langle x, y \rangle - \frac{a}{2}\|y\|^2$, where $\|\cdot\|$ is the Euclidean norm and $a \in \mathbb{R}$ is some constant. Denoting B_a and B_a° the corresponding Moreau conjugacies given by (7), we get that $B_a f = (f + \frac{a}{2}\|\cdot\|^2)^*$ and $B_a^\circ g = -\frac{a}{2}\|\cdot\|^2 + g^*$, hence the properties of B_a can be deduced from those of the Legendre-Fenchel transform. In particular:

COROLLARY 6.6. *Let $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an essentially smooth l.s.c. proper convex function. If f is a l.s.c. function such that $B_a f \leq g$ and $B_a f(x) = g(x)$ for all $x \in \text{idom}(g)$, then $f = B_a^\circ g$.*

Such kernels are useful, for instance, if we want to identify a function f which is semiconvex but not convex. Indeed, if f is semiconvex, there exists $a \in \mathbb{R}$ such that $f + \frac{a}{2}\|\cdot\|^2$ is strictly convex. Hence, $g = B_a f$ satisfies the assumptions of Corollary 6.6. Note that, by standard results of convex analysis, we know that when f is l.s.c. proper and convex, $B_a f$ is the inf-convolution of f^* and $(\frac{a}{2}\|\cdot\|^2)^* = \frac{1}{2a}\|\cdot\|^2$, that is it is the Moreau-Yoshida regularisation of f^* , which explain why $B_a f$ is essentially smooth.

Similar results can be obtained when replacing the kernel b_a by $b'_a(x, y) = -\frac{a}{2}\|x - y\|^2$, with $a \neq 0$.

6.3. ω -Lipschitz Continuous Maps. Let E be a Hausdorff topological vector space and $\omega : E \rightarrow \mathbb{R}^+$ be a continuous subadditive map:

$$\omega(x + y) \leq \omega(x) + \omega(y) \quad \text{for all } x, y \in E,$$

such that $\omega(-x) = \omega(x)$ for all $x \in E$, and $\omega(x) = 0 \Leftrightarrow x = 0$. We say that a function $f : E \rightarrow \mathbb{R}$ is ω -Lipschitz continuous if:

$$|f(y) - f(x)| \leq \omega(y - x) \quad \text{for all } x, y \in E,$$

and we denote by $\text{Lip}_\omega(E)$ the set of ω -Lipschitz continuous functions $f : E \rightarrow \mathbb{R}$. If E is a normed vector space with norm $\|\cdot\|$, then $\omega(x) = a\|x\|^p$ satisfies the above properties for all $a > 0$ and $p \in (0, 1]$, and in that case $\text{Lip}_\omega(E)$ is the set of Hölder continuous functions $f : E \rightarrow \mathbb{R}$ with exponent p and multiplicative constant a .

Take $Y = X = E$ and consider the kernel b given by (9) with $\bar{b}(x, y) = b_\omega(x, y) := -\omega(y - x)$. We denote by B_ω and B_ω° the corresponding Moreau conjugacies given by (7). We have $B_\omega = B_\omega^\circ$. When $\omega(x) = a\|x\|^p$, these Moreau conjugacies were studied by Dolecki and Kurcyusz [DK78, Section 5]. The kernel b is continuous (in the second variable), but b is not coercive, in general, since when $\omega = a\|\cdot\|$ and V is the ball of centre x and radius ε , $b_{x, V}^\alpha(y) = \sup_{z \in V} a(\|y - x\| - \|y - z\|) + \alpha = a\varepsilon + \alpha$. We have:

PROPOSITION 6.7. *Let $g \in \mathcal{G}$. Then $g = B_\omega B_\omega^o g$ if, and only if, either $g \equiv +\infty$, or $g \equiv -\infty$, or $g \in \text{Lip}_\omega(E)$. In that case, we have $B_\omega^o g = -g$.*

PROOF. Let $f \in \mathcal{F}$. If $B_\omega f \not\equiv +\infty$ and $B_\omega f \not\equiv -\infty$, then $f(y) > -\infty$ for all $y \in Y$ and there exists $y \in Y$ such that $f(y) < +\infty$. Moreover, since ω is subadditive, $b_\omega(\cdot, y) - \alpha$ is a ω -Lipschitz continuous function for all $y \in E$ and $\alpha \in \mathbb{R}$. Hence, $B_\omega f$ which is the supremum of a non-empty family of ω -Lipschitz continuous functions, is also ω -Lipschitz continuous, which shows the “only if” part of the proposition.

Conversely, if $g \equiv +\infty$ or $g \equiv -\infty$, then $B_\omega^o g = -g$ and $g = B_\omega B_\omega^o g$. Let $g \in \text{Lip}_\omega(E)$. Since $g(y) - g(x) \leq \omega(y - x)$ for all $x, y \in E$, we deduce:

$$B_\omega^o g(y) = \sup_{x \in E} -\omega(y - x) - g(x) \leq -g(y).$$

Moreover, taking $x = y$ in the supremum, we get that $B_\omega^o g(y) \geq -g(y)$, hence $B_\omega^o g = -g$. Since $B_\omega = B_\omega^o$ and since $-g$ is also in $\text{Lip}_\omega(E)$, it follows, by application of the same argument, that $B_\omega(-g) = g$, hence $B_\omega B_\omega^o g = B_\omega(-g) = g$, which shows the “if” part of the proposition. \square

Proposition 6.7 implies that $B_\omega = B_\omega^o$ is regular, since any map of the form $B_\omega f$ is continuous. Also, $\text{dom}(B_\omega f) = \text{idom}(B_\omega f)$ is equal to E or \emptyset for all $f \in \mathcal{F}$. We also have:

LEMMA 6.8. *A map $f \in \mathcal{F}$ is in \mathcal{F}_c if, and only if, $f + \omega$ has relatively compact finite sublevel sets, which means that $\{y \in E \mid f(y) + \omega(y) \leq \beta\}$ is relatively compact, for all $\beta \in \mathbb{R}$.*

PROOF. By definition, $f \in \mathcal{F}_c$ if, and only if, $b_\omega(x, \cdot) - f$ has relatively compact finite superlevel sets. Since $b_\omega(0, \cdot) = -\omega$, $f \in \mathcal{F}_c$ implies that $f + \omega$ has relatively compact finite sublevel sets. Conversely, if $f + \omega$ has relatively compact finite sublevel sets, then, by subadditivity of ω , $\{y \in E \mid b_\omega(x, y) - f(y) \geq \beta\} \subset \{y \in E \mid f(y) + \omega(y) \leq \omega(x) - \beta\}$ is relatively compact for all $\beta \in \mathbb{R}$ and $x \in E$. \square

The condition of Lemma 6.8 holds in particular when $E = \mathbb{R}^n$, $\omega = a\|\cdot\|$ for some norm on E and f is Lipschitz continuous with a Lipschitz constant $b < a$. We can apply Corollaries 3.10 and 4.9, and Theorem 4.6 to any map g such that $f = B_\omega^o g$ satisfies the condition of Lemma 6.8. Using Proposition 6.7, we obtain:

COROLLARY 6.9. *Let $g \in \text{Lip}_\omega(E)$ be such that $g - \omega$ has relatively compact finite superlevel sets, then $\{(\partial^o g)^{-1}(y)\}_{y \in E}$ is a covering of E . If*

$$(50) \quad |g(y) - g(x)| < \omega(y - x) \quad \text{for all } x, y \in E \text{ such that } x \neq y,$$

then $(f \in \mathcal{F} \text{ and } B_\omega f = g) \implies f = -g$.

PROOF. The first assertion follows from Corollary 3.10 and Lemma 6.8. We shall prove that under (50), the covering is algebraically (hence topologically) minimal, which will imply the last assertion of the corollary, by using Corollary 4.9. Indeed, if $x, y \in E$, then

$$(51) \quad x \notin \bigcup_{z \in E \setminus \{y\}} (\partial^o g)^{-1}(z) \iff z \notin \partial^o g(x) \quad \forall z \in E \setminus \{y\} \iff \partial^o g(x) \subset \{y\}.$$

Since $g \in \text{Lip}_\omega(E)$, we get $B_\omega^\circ g = -g$, and, by (11b),

$$\begin{aligned}\partial^\circ g(x) &= \{z \in E \mid B_\omega^\circ g(z) = b_\omega(x, z) - g(x)\} \\ &= \{z \in E \mid g(z) - g(x) = \omega(z - x)\} \\ &= \{z \in E \mid g(z) - g(x) \geq \omega(z - x)\}.\end{aligned}$$

Then $x \in \partial^\circ g(x)$ for all $x \in E$, and

$$(52) \quad \partial^\circ g(x) \subset \{y\} \iff x = y \text{ and } g(z) - g(y) < \omega(z - y) \quad \forall z \in E \setminus \{y\}.$$

Using (51), (52) and Definition 4.1, we get that the covering $\{(\partial^\circ g)^{-1}(y)\}_{y \in E}$ of E is algebraically minimal if, and only if, for all $y \in E$, $g(z) - g(y) < \omega(z - y)$ for all $z \in E \setminus \{y\}$, which (by symmetry) is equivalent to Condition (50). \square

6.4. L.s.c. Maps bounded from below. Let $(E, \|\cdot\|)$ be a normed space, fix a constant $p > 0$, take $Y = E$, $X = E \times (0, +\infty)$ and consider the kernel b given by (9) with $\bar{b}(x, y) = -x''\|y - x'\|^p$, with $x = (x', x'')$, $x' \in E$, and $x'' \in (0, +\infty)$. We denote by B and B° the corresponding Moreau conjugacies given by (7). These Moreau conjugacies were studied in [DK78, Section 4]. When $p = 1$, B is used in [Sam02] to define a (semi-)distance on the set of quasi-continuous functions from E to \mathbb{R} . When $p \leq 1$, $f \in \mathcal{F}$, and $x = (x', x'') \in E \times (0, +\infty)$, $Bf(x) = B_\omega f(x')$ where B_ω is given as in Section 6.3 with $\omega = x''\|\cdot\|^p$, and the results of this latter section show that B is injective on the set of Hölder continuous functions with exponent p . When $E = \mathbb{R}^n$, $\|\cdot\|$ is the Euclidean norm, $p = 2$, $f \in \mathcal{F}$, and $x = (x', x'') \in E \times (0, +\infty)$, $Bf(x) = -x''\|x'\|^2 + B_{2x''}f(2x''x')$ where $B_{2x''}$ is given as in Section 6.2. The results of this latter section show that B is injective on the set of semiconvex maps. Proposition 6.11 below shows that indeed, for all $p > 0$, B is injective on a large set of l.s.c. functions. We first prove some preliminary results.

LEMMA 6.10. *Let $f \in \mathcal{F}$. Then either $Bf \equiv +\infty$, or $Bf \equiv -\infty$, or there exists $a \geq 0$ such that*

$$(53) \quad E \times (a, +\infty) = \text{idom}(Bf) \subset \text{dom}(Bf) \subset E \times [a, +\infty).$$

More precisely, for all $x = (x', x'')$, $z = (z', z'') \in X$ such that $x'' > z''$, we have

$$(54a) \quad Bf(x) \leq Bf(z) + K(x, z),$$

$$(54b) \quad \text{with } K(x, z) := \begin{cases} \left((z'')^{\frac{1}{1-p}} - (x'')^{\frac{1}{1-p}} \right)^{1-p} \|x' - z'\|^p & \text{when } p > 1, \\ z''\|x' - z'\|^p & \text{when } p \leq 1. \end{cases}$$

PROOF. Assume that $Bf \not\equiv +\infty$, and $Bf \not\equiv -\infty$. Then there exists $y \in Y$ such that $f(y) < +\infty$, which implies that $Bf(x) > -\infty$ for all $x \in X$, hence $\text{dom}(Bf) \neq \emptyset$. Assume first that (54) is proved. Let

$$a = \inf\{x'' \in (0, +\infty) \mid \exists x' \in E \text{ such that } Bf(x', x'') < +\infty\}.$$

Then $\text{dom}(Bf) \subset E \times [a, +\infty)$, and since $\text{idom}(Bf)$ is open and included in $\text{dom}(Bf)$, we get that $\text{idom}(Bf) \subset E \times (a, +\infty)$. Conversely, let $x_0 = (x'_0, x''_0) \in E \times (a, +\infty)$. By definition of a , there exists $z = (z', z'') \in E \times (a, x''_0)$ such that $Bf(z) < +\infty$. Let $\varepsilon = \frac{x''_0 - z''}{2} > 0$ and consider the neighbourhood V of x_0 given by $V = \{(x', x'') \in X \mid \|x' - x'_0\| \leq \varepsilon \text{ and } |x'' - x''_0| \leq \varepsilon\}$. Then $x'' - z'' \geq \varepsilon$ and $\|x' - z'\| \leq \|x'_0 - z'\| + \varepsilon$, for all $(x', x'') \in V$. Using (54), we obtain:

$$\sup_{x \in V} Bf(x) \leq Bf(z) + \sup_{x \in V} K(x, z),$$

and

$$\sup_{x \in V} K(x, z) \leq \begin{cases} \left((z'')^{\frac{1}{1-p}} - (z'' + \varepsilon)^{\frac{1}{1-p}} \right)^{1-p} (\|x'_0 - z'\| + \varepsilon)^p & \text{when } p > 1, \\ z''(\|x'_0 - z'\| + \varepsilon)^p & \text{when } p \leq 1, \end{cases}$$

hence $x_0 \in \text{idom}(Bf)$, which finishes the proof of (53).

Let us now prove (54). Let $x = (x', x'')$, $z = (z', z'') \in X$ be such that $x'' > z''$. Using the definition of Bf , we deduce that $Bf(x) \leq Bf(z) + K_0(x, z)$, where

$$\begin{aligned} K_0(x, z) &= \sup_{y \in E} (-x''\|y - x'\|^p + z''\|y - z'\|^p) \\ &\leq \sup_{y \in E} (-x''\|y - x'\|^p + z''(\|y - x'\| + \|x' - z'\|)^p) \\ (55) \quad &\leq \sup_{\rho \geq 0} (-x''\rho^p + z''(\rho + \|x' - z'\|)^p). \end{aligned}$$

Computing the supremum in (55), we obtain that $K_0(x, z) \leq K(x, z)$ with K given by (54b), which shows (54). \square

PROPOSITION 6.11. *Let $f \in \mathcal{F}$. Then $f = B^\circ Bf$ if, and only if, $f \equiv -\infty$, or there exists $a > 0$ such that $f + a\|\cdot\|^p$ is bounded from below. Moreover, $\text{dom}(Bf) \neq \emptyset \implies f = B^\circ Bf$.*

PROOF. If $f = B^\circ Bf$ and $f \not\equiv -\infty$, then there exists $x \in X$ such that $Bf(x) < +\infty$. By Lemma 6.10, either $Bf \equiv -\infty$, or (53) holds. In the first case, $f \equiv +\infty$, thus $f + a\|\cdot\|^p$ is bounded from below, for all $a > 0$. In the second case, there exists $a \geq 0$ such that $\text{idom}(Bf) = E \times (a, +\infty)$. Taking $a' > a$, we get that $Bf(0, a') < +\infty$, which means that $f + a'\|\cdot\|^p$ is bounded from below.

Conversely, if $f \equiv -\infty$ or $f \equiv +\infty$, then $f = B^\circ Bf$. Assume that $f \in \mathcal{F}$ is such that $f + a\|\cdot\|^p$ is bounded from below, for some $a > 0$, and that $f \not\equiv +\infty$. Then, $Bf(0, a) < +\infty$ and $Bf(x) > -\infty$ for all $x \in X$, which shows that $\text{dom}(Bf) \neq \emptyset$. It remains to prove the last assertion of the proposition, that we shall derive from a result of [DK78]. Recall that if Φ is a set of real valued functions on E , a function f is Φ -convex if it can be written as a pointwise supremum of a possibly infinite family of elements of Φ . Let us take for Φ the set of functions from E to \mathbb{R} of the form $y \mapsto -x''\|y - x'\|^p + r$ with $x'' > 0$, $r \in \mathbb{R}$ and $x' \in E$. When $\text{dom}(Bf) \neq \emptyset$, f is bounded from below by an element of Φ . Then, by [DK78, Theorem 4.2], f is Φ -convex, which implies that $f = B^\circ Bf$. \square

In order to deduce covering properties as in Section 4, we need to show some properties of b and B . First, it is clear that b is continuous (in the second variable). The following result may be compared with [DK78, Lemma 4.4]

LEMMA 6.12. *When $E = \mathbb{R}^n$, the kernel b is coercive.*

PROOF. Let $x \in E$, $\alpha \in \mathbb{R}$ and V be a neighbourhood of x . The map $b_{x, V}^\alpha$ defined in (13) satisfies:

$$b_{x, V}^\alpha(y) = \sup_{z \in V} \bar{b}(z, y) - \bar{b}(x, y) + \alpha = \sup_{(z', z'') \in V} -z''\|y - z'\|^p + x''\|y - x'\|^p + \alpha.$$

Let $\varepsilon > 0$ be such that $V \supset \{(z', z'') \in X \mid \|z' - x'\| \leq \varepsilon, |z'' - x''| \leq \varepsilon\}$. We get that $b_{x, V}^\alpha(y) \geq \varepsilon\|y - x'\|^p + \alpha$, hence $b_{x, V}^\alpha$ has bounded sublevel sets. When $E = \mathbb{R}^n$, this implies that $b_{x, V}^\alpha$ has relatively compact sublevel sets, hence b is coercive. \square

LEMMA 6.13. *B is regular.*

PROOF. Let $f \in \mathcal{F}$ and $g = Bf$. Then g is l.s.c. as the supremum of continuous maps. To show that g is quasi-continuous on its domain it suffices to prove that $g = \text{lsc}(\text{usc}(g))$ where lsc and usc envelopes are applied to the restrictions to $\text{dom}(g)$. Moreover, since g is l.s.c., we get $g \leq \text{lsc}(\text{usc}(g))$, hence it is sufficient to prove that $g \geq \text{lsc}(\text{usc}(g))$. This is true if $g \equiv +\infty$ or $g \equiv -\infty$. Otherwise, (53) and (54) hold and $\text{idom}(g) = X$. In particular, for all fixed $z = (z', z'') \in \text{dom}(g)$, $E \times (z'', +\infty)$ is open and included in $\text{dom}(g)$, $g(x) \leq g(z) + K(x, z)$ for all $x \in E \times (z'', +\infty)$, and $x \mapsto K(x, z)$ is continuous on $E \times (z'', +\infty)$. Hence

$$(56) \quad \text{usc}(g)(x) \leq g(z) + K(x, z) \text{ for all } z = (z', z'') \in \text{dom}(g), x \in E \times (z'', +\infty).$$

Since $\text{idom}(g)$ is the interior of $\text{dom}(g)$, for all $x = (x', x'') \in \text{idom}(g)$, there exists $\varepsilon > 0$ such that $z = (x', x'' - \varepsilon) \in \text{dom}(g)$. Hence, using (56), we get that $\text{usc}(g)(x) \leq g(z) + K(x, z) = g(z)$. It follows that

$$(57) \quad \text{usc}(g)(x) \leq \lim_{\varepsilon \rightarrow 0^+} g(x', x'' - \varepsilon).$$

Moreover, it is clear that for all fixed $x' \in E$, $x'' \in (0, +\infty) \mapsto g(x', x'')$ is a nonincreasing l.s.c. proper convex map. Since $(0, +\infty)$ is one dimensional, this implies in particular that this map is continuous on its domain (see [Roc70]). Therefore, it follows from (57) that $\text{usc}(g)(x) \leq g(x)$ for all $x \in \text{idom}(g)$. This shows that g is continuous in the interior of its domain. Now, by (53), if $x = (x', x'') \in \text{dom}(g)$, $(x', x'' + \varepsilon) \in \text{idom}(g)$ for all $\varepsilon > 0$, and since $x'' \in (0, +\infty) \mapsto g(x', x'')$ is continuous on its domain, we get

$$\text{lsc}(\text{usc}(g))(x) \leq \liminf_{\varepsilon \rightarrow 0^+} \text{usc}(g)(x', x'' + \varepsilon) = \liminf_{\varepsilon \rightarrow 0^+} g(x', x'' + \varepsilon) = g(x),$$

which finishes the proof of $\text{lsc}(\text{usc}(g)) \leq g$. \square

COROLLARY 6.14. *Assume that $E = \mathbb{R}^n$. Let $g \in \mathcal{G}$ such that $\text{dom}(g) \neq \emptyset$ and $g = BB^{\circ}g$. Then $(f \in \mathcal{F} \text{ and } Bf = g) \implies f = B^{\circ}g$. Moreover $\{(\partial^{\circ}g)^{-1}(y)\}_{y \in \text{dom}(B^{\circ}g)}$ is a topological minimal covering of $\text{idom}(g)$. Finally, if $B^{\circ}g$ is quasi-continuous on its domain, then $(Bf \leq g \text{ and } Bf = g \text{ on } \text{idom}(g)) \implies f = B^{\circ}g$.*

PROOF. The first assertion of the corollary follows from the last assertion of Proposition 6.11, since $\text{dom}(g) \neq \emptyset$. This shows that Problem (\mathcal{P}) has a unique solution. Since b is continuous in the second variable and $\text{dom}(g) \neq \emptyset$, Assumption (A5)' holds. Moreover, b is coercive (by Lemma 6.12), B is regular (by Lemma 6.13), Condition (C) holds, and $\text{dom}(g)$ is included in the closure of $\text{idom}(g)$ (by (53)). Hence, applying Theorem 4.10 in Case (3), we obtain the second assertion of the corollary. Finally, applying the implication (33) \Rightarrow (32) in Theorem 4.7 to $X' = \text{idom}(g)$, we obtain the last assertion of the corollary. \square

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